Journal of Algebra 321 (2009) 2970–3025

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Socles and radicals of Mackey functors

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ARTICLE INFO

Article history:
Received 14 August 2008
Available online 26 February 2009
Communicated by Michel Broué

Keywords:
Mackey functor
Mackey algebra
Maximal subfunctor
Simple subfunctor
Socle
Radical
Loewy length
Brauer quotient
Restriction kernel
Primordial subgroup
Burnside functor

ABSTRACT

We study the socle and the radical of a Mackey functor M for a finite group G over a field $\mathbb{K}$ (usually, of characteristic $p > 0$). For a subgroup H of G, we construct bijections between some classes of the simple subfunctors of M and some classes of the simple $\mathbb{K}\overline{N}_G(H)$-submodules of $M(H)$. We relate the multiplicity of a simple Mackey functor $S^G_{H,V}$ in the socle of M to the multiplicity of V in the socle of a certain $\mathbb{K}\overline{N}_G(H)$-submodule of $M(H)$. We also obtain similar results for the maximal subfunctors of M. We then apply these general results to some special Mackey functors for G, including the functors obtained by inducing or restricting a simple Mackey functor, Mackey functors for a p-group, the fixed point functor, and the Burnside functor $\mathrm{B}^G_{\mathbb{K}}$. For instance, we find the first four top factors of the radical series of $\mathrm{B}^G_{\mathbb{K}}$ for a p-group G, and assuming further that G is an abelian p-group we find the radical series of $\mathrm{B}^G_{\mathbb{K}}$.

1. Introduction

The purpose of the present paper is to study the socle and the radical of a Mackey functor for a finite group, a notion that was introduced by J.A. Green [4] and A. Dress [3]. The theory of Mackey functors was developed mainly by J. Thévenaz and P. Webb in [10,11]. In particular, they showed that Mackey functors for a finite group may be seen as modules of a finite dimensional algebra.

Let G be a finite group and H be a subgroup of G, and let $\mathbb{K}$ be a field (usually, of characteristic $p > 0$). After recalling some preliminary results in Section 2, we first study the socle and the radical of a Mackey functor over $\mathbb{K}$ obtained by restricting or inducing a simple Mackey functor. For instance, we observe in Section 3 that if M is a Mackey functor for G of the form $\uparrow^G_H S^H_{K,W}$ for some simple Mackey functor $S^H_{K,W}$ for H then the socle and the radical of M can be determined from the socle and

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0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2009.02.012

the radical of the $\mathbb{K}\overline{N}_G(K)$-module $\uparrow_{\overline{N}_H(K)}^{\overline{N}_G(K)} W$. The main reason for this observation is the property of M that the evaluation of any nonzero subfunctor of M at H is nonzero. We next begin to study the socle and the radical of a Mackey functor that may not satisfy the above mentioned property.

Let M be a Mackey functor for G. In Section 4 we construct a bijective correspondence between the maximal subfunctors of M whose simple quotients have H as minimal subgroups and the maximal $\mathbb{K}\overline{N}_G(H)$-submodules of $\overline{M}(H)$ satisfying some certain conditions where $\overline{M}(H)$ denotes the Brauer quotient of $M(H)$. We further study and refine this bijection in Section 5. We also study the simple subfunctors of M having H as minimal subgroups and relate them to the simple $\mathbb{K}\overline{N}_G(H)$-submodules of $\underline{M}(H)$ satisfying some certain conditions where $\underline{M}(H)$ denotes the restriction kernel of $M(H)$. For instance, we show in Section 5 that, given a simple $\mathbb{K}\overline{N}_G(H)$-module U, the multiplicity of the simple Mackey functor $S_{H,U}^G$ in the socle of M is equal to the multiplicity of U in the socle of the following $\mathbb{K}\overline{N}_G(H)$-module

$$\bigcap_{X/H}\left\{x\in \underline{M}(H)\colon \left(\sum_{gH\subseteq X} c_H^g\right)x=0 \implies t_H^X(x)=0\right\}$$

where X/H ranges over all nontrivial p-subgroups of $N_G(H)/H$.

We devote Section 6 to studying Mackey functors satisfying some extreme conditions such as being a functor for a p-group, having a unique (up to G-conjugacy) maximal primordial subgroup, having a unique simple subfunctor, being uniserial. For these special functors we give some consequences of the results we obtained in the previous sections. To mention one of them, we show that the primordial subgroups of a uniserial Mackey functor form a chain with respect to the subgroup conjugacy relation.

Our aim in Section 7 is to illustrate some applications of the general results in previous sections. The main object we apply our results is the Burnside functor $\mathrm{B}_{\mathbb{K}}^G$ for G, which has a special importance for the category of Mackey functors because any projective indecomposable Mackey functor for G is a direct summand of the functor $\uparrow_H^G \mathrm{B}_{\mathbb{K}}^H$ for some subgroup H of G. We first describe the maximal subfunctors of $\mathrm{B}_{\mathbb{K}}^G$. We find some results about simple Mackey functors appearing in the factors of the radical series of $\mathrm{B}_{\mathbb{K}}^G$. For example, we show that $S_{1,\mathbb{K}}^G$, whose multiplicity as a composition factor of $\mathrm{B}_{\mathbb{K}}^G$ is 1, appears in J_m/J_{m+1} where J_m is the mth term of the radical series of $\mathrm{B}_{\mathbb{K}}^G$ and p^m is the order of a Sylow p-subgroup of G. Assuming that G is a p-group we find the first four top factors of the radical series of $\mathrm{B}_{\mathbb{K}}^G$. Assuming further that G is an abelian p-group, we find the radical series of $\mathrm{B}_{\mathbb{K}}^G$. After that we try to study the socle series of $\mathrm{B}_{\mathbb{K}}^G$ and observe that this is much harder than the study of the radical series even when G is an abelian p-group. For the socle series we only obtain some limited results.

There are two works related to this paper we want to mention. The first one is Webb [12] in which two kinds of filtration of a Mackey functor whose factors are related to the Brauer quotients and the restriction kernels are constructed. The second one is Nicollerat [7] in which the socle of a projective Mackey functor for a p-group is studied. In particular, the socle of $\mathrm{B}_{\mathbb{K}}^G$ is determined completely in [7] for some classes of abelian p-groups.

We finally want to explain our notations. Let H and K be subgroups of G. By the notation $HgK\subseteq G$ we mean that g ranges over a complete set of representatives of double cosets of (H,K) in G. We write $\overline{N}_G(H)$ for the quotient group $N_G(H)/H$ where $N_G(H)$ is the normalizer of H in G. For a module V of an algebra we denote by $\mathrm{Soc}(V)$ and $\mathrm{Jac}(V)$ the socle and the radical of V, respectively. Most of our other notations are standard and tend to follow [10,11].

Throughout, G is a finite group, $\mathbb{K}$ is a field. We consider only finite dimensional Mackey functors.

2. Preliminaries

In this section, we briefly summarize some crucial material on Mackey functors. For the proofs, see Thévenaz and Webb [10,11]. Recall that a Mackey functor for G over a commutative unital ring R is such that, for each subgroup H of G, there is an R-module $M(H)$; for each pair $H,K\leqslant G$ with $H\leqslant K$, there are R-module homomorphisms $r_H^K : M(K) \to M(H)$ called the restriction map and $t_H^K : M(H) \to M(K)$ called the transfer map or the trace map; for each $g \in G$, there is an R-module homomorphism $c_H^g : M(H) \to M({}^gH)$ called the conjugation map. The following axioms must be satisfied for any $g, h \in G$ and $H, K, L \leqslant G$ [1,4,10,11].

(M$_1$) if $H \leqslant K \leqslant L$, $r_H^L = r_H^K r_K^L$ and $t_H^L = t_K^L t_H^K$; moreover $r_H^H = t_H^H = \mathrm{id}_{M(H)}$;
(M$_2$) $c_K^{gh} = c_{{}^hK}^g c_K^h$;
(M$_3$) if $h \in H$, $c_H^h : M(H) \to M(H)$ is the identity;
(M$_4$) if $H \leqslant K$, $c_H^g r_H^K = r_{{}^gH}^{{}^gK} c_K^g$ and $c_K^g t_H^K = t_{{}^gH}^{{}^gK} c_H^g$;
(M$_5$) (Mackey Axiom) if $H \leqslant L \geqslant K$, $r_H^L t_K^L = \sum_{HgK \subseteq L} t_{H \cap {}^gK}^H r_{H \cap {}^gK}^{{}^gK} c_K^g$.

Another possible definition of Mackey functors for G over R uses the Mackey algebra $\mu_R(G)$ [1,11]: $\mu_{\mathbb{Z}}(G)$ is the algebra generated by the elements r_H^K, t_H^K, and c_H^g, where H and K are subgroups of G such that $H \leqslant K$, and $g \in G$, with the relations (M$_1$)–(M$_7$).

(M$_6$) $\sum_{H \leqslant G} t_H^H = \sum_{H \leqslant G} r_H^H = 1_{\mu_{\mathbb{Z}}(G)}$;
(M$_7$) any other product of r_H^K, t_H^K and c_H^g is zero.

A Mackey functor M for G, defined in the first sense, gives a left module $\widetilde{M}$ of the associative R-algebra $\mu_R(G) = R \otimes_{\mathbb{Z}} \mu_{\mathbb{Z}}(G)$ defined by $\widetilde{M} = \bigoplus_{H \leqslant G} M(H)$. Conversely, if $\widetilde{M}$ is a $\mu_R(G)$-module then $\widetilde{M}$ corresponds to a Mackey functor M in the first sense, defined by $M(H) = t_H^H \widetilde{M}$, the maps t_H^K, r_H^K, and c_H^g being defined as the corresponding elements of the $\mu_R(G)$. Moreover, homomorphisms and subfunctors of Mackey functors for G are $\mu_R(G)$-module homorphisms and $\mu_R(G)$-submodules, and conversely.

Theorem 2.1. *(See [11].) Letting H and Krun over all subgroups of G, letting g run over representatives of the double cosets $HgK \subseteq G$, and letting J runs over representatives of the conjugacy classes of subgroups of $H^g \cap K$, then $t_{{}^gJ}^H c_J^g r_J^K$ comprise, without repetition, a free R-basis of $\mu_R(G)$.*

Let M be a Mackey functor for G over R. A subgroup H of G is called a minimal subgroup of M if $M(H) \neq 0$ and $M(K) = 0$ for every subgroup K of H with $K \neq H$. Given a simple Mackey functor M for G over R, there is a unique, up to G-conjugacy, a minimal subgroup H of M. Moreover, for such an H the $R\overline{N}_G(H)$-module $M(H)$ is simple, where the $R\overline{N}_G(H)$-module structure on $M(H)$ is given by $gH.x = c_H^g(x)$, see [10].

Theorem 2.2. *(See [10].) Given a subgroup $H \leqslant G$ and a simple $R\overline{N}_G(H)$-module V, then there exists a simple Mackey functor $S_{H,V}^G$ for G, unique up to isomorphism, such that H is a minimal subgroup of $S_{H,V}^G$ and $S_{H,V}^G(H) \cong V$. Moreover, up to isomorphism, every simple Mackey functor for G has the form $S_{H,V}^G$ for some $H \leqslant G$ and simple $R\overline{N}_G(H)$-module V. Two simple Mackey functors $S_{H,V}^G$ and $S_{H',V'}^G$ are isomorphic if and only if, for some element $g \in G$, we have $H' = {}^gH$ and $V' \cong c_H^g(V)$.*

We now recall the definitions of restriction, induction and conjugation for Mackey functors [1,8,10, 11]. Let M and T be Mackey functors for G and H, respectively, where $H \leqslant G$.

The restricted Mackey functor $\downarrow_H^G M$ is the $\mu_R(H)$-module $1_{\mu_R(H)} M$ so that $(\downarrow_H^G M)(X) = M(X)$ for $X \leqslant H$.

For $g \in G$, the conjugate Mackey functor $|_H^g T = {}^gT$ is the $\mu_R({}^gH)$-module T with the module structure given for any $x \in \mu_R({}^gH)$ and $t \in T$ by $x.t = (\gamma_{g^{-1}} x \gamma_g)t$, where γ_g is the sum of all c_X^g with X ranging over subgroups of G. Therefore, $(|_H^g T)({}^gX) = T(X)$ for all $X \leqslant H$ and the maps $\tilde{t}, \tilde{r}, \tilde{c}$ of $|_H^g T$ satisfy $\tilde{t}_B^A = t_{B^g}^{A^g}$, $\tilde{r}_B^A = r_{B^g}^{A^g}$, and $\tilde{c}_A^x = c_{A^g}^{x^g}$ where t, r, c are the maps of T. Obviously, one has $|_L^g S_{H,V}^L \cong S_{{}^gH, c_H^g(V)}^{{}^gL}$.

The induced Mackey functor $\uparrow_H^G T$ is the $\mu_R(G)$-module $\mu_R(G)1_{\mu_R(H)} \otimes_{\mu_R(H)} T$, where $1_{\mu_R(H)}$ denotes the unity of $\mu_R(H)$. It may be useful to express the $\mu_R(G)$-module $\uparrow_H^G T$ as a Mackey functor in the first sense which is the context of the next result. By the axioms (M_1)–(M_7) defining the Mackey algebra, it can be seen easily that for any $K \leqslant G$, we have

$$t_K^K \mu_R(G) 1_{\mu_R(H)} = \bigoplus_{KgH \subseteq G} c_{K^g}^g t_{H \cap K^g}^{K^g} \mu_R(H).$$

Therefore

$$\left(\uparrow_H^G T\right)(K) = t_K^K\left(\mu_R(G) 1_{\mu_R(H)} \otimes_{\mu_R(H)} T\right) = \bigoplus_{KgH \subseteq G} c_{K^g}^g t_{H \cap K^g}^{K^g} \otimes_{\mu_R(H)} t_{H \cap K^g}^{H \cap K^g} T.$$

The following result is clear now.

Proposition 2.3. *(See [8,10].) Let H be a subgroup of G and T be a Mackey functor for H. Then for any subgroup K of G,*

$$\left(\uparrow_H^G T\right)(K) \cong \bigoplus_{KgH \subseteq G} T\left(H \cap K^g\right)$$

as R-modules. In particular, if $T(X) \neq 0$ for some subgroup X of H then $(\uparrow_H^G T)(X) \neq 0$.

The induced Mackey functor $\uparrow_H^G T$ can also be defined by giving its values on subgroups K of G as the R-modules in the right-hand side of the isomorphism in 2.3, and by giving its maps t, r, c in terms of the maps of T. See [8,10].

Proposition 2.4. *Let $H \leqslant K \leqslant G$ and let W be a simple $R\overline{N}_K(H)$-module. Then*:

(1) [15, Lemma 7.2] *We have the direct sum decomposition $t_H^H \mu_R(G) t_H^H = A_H \oplus I_H$ where A_H is a unital subalgebra of $t_H^H \mu_R(G) t_H^H$ isomorphic to $R\overline{N}_G(H)$ (via the map $c_H^g \mapsto gH$) and I_H is a two sided ideal of $t_H^H \mu_R(G) t_H^H$ with the R-basis consisting of the elements of the form $t_{{}^gJ}^H c_J^g r_J^H$ where $J \neq H$.*

(2) [15, Lemma 6.12] *$(\uparrow_K^G S_{H,W}^K)(H) \cong \uparrow_{\overline{N}_K(H)}^{\overline{N}_G(H)} W$ as $R\overline{N}_G(H)$-modules.*

Theorem 2.5. *(See [8].) Let H be a subgroup of G. Then $\uparrow_H^G$ is both left and right adjoint of $\downarrow_H^G$.*

Given $H \leqslant G \geqslant K$ and a Mackey functor M for K over R, the following is the Mackey decomposition formula for Mackey algebras, which can be found in [11],

$$\downarrow_H^L \uparrow_K^L M \cong \bigoplus_{HgK \subseteq L} \uparrow_{H \cap {}^gK}^H \downarrow_{H \cap {}^gK}^{{}^gK} |_K^g M.$$

We finally recall some facts from [10] about inflated Mackey functors. Let N be a normal subgroup of G. Given a Mackey functor $\widetilde{M}$ for G/N, we define a Mackey functor $M = \mathrm{Inf}_{G/N}^G \widetilde{M}$ for G, called the inflation of $\widetilde{M}$, as $M(K) = \widetilde{M}(K/N)$ if $K \geqslant N$ and $M(K) = 0$ otherwise. The maps t_H^K, r_H^K, c_H^g of M are zero unless $N \leqslant H \leqslant K$ in which case they are the maps $\tilde{t}_{H/N}^{K/N}, \tilde{r}_{H/N}^{K/N}, \tilde{c}_{H/N}^{gN}$ of $\widetilde{M}$. Evidently, one has $\mathrm{Inf}_{G/N}^G S_{H/N,V}^{G/N} \cong S_{H,V}^G$.

Given a Mackey functor M for G we define Mackey functors $L^+{}_{G/N}M$ and $L^-{}_{G/N}M$ for G/N as follows:

$$\left(L^{+}{}_{G/N}M\right)(K/N)=M(K)/\sum_{J\leqslant K:J\ngeqslant N} t_J^K\left(M(J)\right),$$

$$\left(L^{-}{}_{G/N}M\right)(K/N)=\bigcap_{J\leqslant K:J\ngeqslant N}\operatorname{Ker} r_J^K.$$

The maps on these two new functors come from those on M. They are well defined because the maps on M preserve the sum of images of traces and the intersection of kernels of restrictions, see [10].

Theorem 2.6. *(See [10].) For any normal subgroup N of G, $L^{+}{}_{G/N}$ is a left adjoint of $\operatorname{Inf}_{G/N}^{G}$ and $L^{-}{}_{G/N}$ is a right adjoint of $\operatorname{Inf}_{G/N}^{G}$.*

Theorem 2.7. *(See [10].) For any simple $\mu_{\mathbb{K}}(G)$-module $S_{H,V}^{G}$, we have*

$$S_{H,V}^{G}\cong\uparrow_{N_G(H)}^{G}\operatorname{Inf}_{N_G(H)/H}^{N_G(H)}S_{1,V}^{\overline{N}_G(H)}\cong\uparrow_{N_G(H)}^{G}S_{H,V}^{N_G(H)}.$$

3. Induction and restriction of simple functors

Our main aim in this section is to study the socle and the radical of a Mackey functor obtained by restricting or inducing a simple functor.

Let T be a Mackey functor for a subgroup K of G. Relating $\operatorname{Soc}(\uparrow_K^G T)$ to $\operatorname{Soc}(T)$ may require finding a relation between the minimal subgroups of the functors $\uparrow_K^G T$ and T. It is not true in general that any minimal subgroup of T is also a minimal subgroup of $\uparrow_K^G T$. For instance, if the subgroup K have subgroups A and B satisfying $A <_G B$ but $A \nless_K B$ then we may take $T=S_{A,\mathbb{K}}^{K}\oplus S_{B,\mathbb{K}}^{K}$ so that, by the explicit description of an induced functor given in 2.3, the minimal subgroup B of T is not a minimal subgroup of $\uparrow_K^G T$. However if T is simple then it is clear by 2.3 that the minimal subgroups of $\uparrow_K^G T$ are precisely the G-conjugates of the minimal subgroups of T. Thus part (6) of [15, Lemma 6.1] is true only when T is simple, and must be corrected as the first part of the following result. However the results of [15] depending on it remain true because they made use of it when T is simple.

Lemma 3.1. *Let K be a subgroup of G.*

(1) *If T is a $\mu_{\mathbb{K}}(K)$-module, then the minimal subgroups of $\uparrow_K^G T$ are precisely the smallest elements (with respect to $\subseteq$) of the set of all G-conjugates of the minimal subgroups of T.*
(2) *If M be a $\mu_{\mathbb{K}}(G)$-module, then the minimal subgroups of $\downarrow_K^G M$ are precisely the minimal subgroups of M that are contained in K.*

Proof. Part (2) is obvious, and part (1) may be proved easily by using the explicit description of the induced functors given in 2.3. □

Lemma 3.2. *Let K be a subgroup of G. Then:*

(1) *For any simple $\mu_{\mathbb{K}}(K)$-module $S_{H,W}^{K}$, the minimal subgroups of any nonzero $\mu_{\mathbb{K}}(G)$-submodule of $\uparrow_K^G S_{H,W}^{K}$ are precisely the G-conjugates of H.*
(2) *For any simple $\mu_{\mathbb{K}}(G)$-module $S_{L,V}^{G}$ with $L\leqslant_G K$, any minimal subgroup of any nonzero $\mu_{\mathbb{K}}(K)$-submodule of $\downarrow_K^G S_{L,V}^{G}$ is a G-conjugate of L.*

Proof. (1) Let M be a nonzero $\mu_{\mathbb{K}}(G)$-submodule of $\uparrow_K^G S_{H,W}^{K}$, and let X be a minimal subgroup of M. As $(\uparrow_K^G S_{H,W}^{K})(X)\neq 0$, we can find a minimal subgroup of $\uparrow_K^G S_{H,W}^{K}$ contained in X. Part (1) of 3.1 implies that $H\leqslant_G X$. From the adjointness of the pair $(\downarrow_K^G,\uparrow_K^G)$ we see the existence of a

$\mu_{\mathbb{K}}(K)$-epimorphism $\downarrow_K^G M \to S_{H,W}^K$. This implies that $M(H) \neq 0$. Since X is a minimal subgroup of the Mackey functor M for G, we conclude that $X =_G H$.

(2) Let T be a nonzero $\mu_{\mathbb{K}}(K)$-submodule of $\downarrow_K^G S_{L,V}^G$, and let Y be a minimal subgroup of T. Then $(\downarrow_K^G S_{L,V}^G)(Y) \neq 0$ implying that $L \leqslant_G Y$.

Let T' denote the functor $\downarrow_Y^K T$. Then T' is a nonzero $\mu_{\mathbb{K}}(Y)$-submodule of $\downarrow_Y^G S_{L,V}^G$. From the adjointness of the pair $(\uparrow_Y^G, \downarrow_Y^G)$ we see the existence of a $\mu_{\mathbb{K}}(G)$-epimorphism $\uparrow_Y^G T' \to S_{L,V}^G$. This implies that $(\uparrow_Y^G T')(L) \neq 0$ from which we see by 2.3 that $0 \neq T'(Y \cap L^g) = T(Y \cap L^g)$ for some $g \in G$. Since Y is a minimal subgroup of T we conclude that $Y \leqslant Y \cap L^g$. $\square$

The above lemma is a combination of [15, Lemma 6.13] and [13, Remark 3.1].

For an algebra A and an idempotent e of A, there are some well-known relations between the module categories of the algebras A and eAe. In particular, the map $S \mapsto eS$ define a bijective correspondence between the isomorphism classes of simple A-modules not annihilated by e and the isomorphism classes of simple eAe-modules. Most of these can be found in [5, pp. 83–87] from which the following lemma follows easily. For any subset X of the A-module V we denote by AX the A-submodule of V generated by X.

Lemma 3.3. *Let A be a finite dimensional $\mathbb{K}$-algebra and let e be a nonzero idempotent of A. If V is a nonzero A-module having no nonzero A-submodule annihilated by e, then:*

(1) *The maps $S \to eS$ and $AT \leftarrow T$ define a bijective correspondence between the simple A-submodules of V and the simple eAe-submodules of eV.*
(2) $\mathrm{Soc}_{eAe}(eV) = e\,\mathrm{Soc}_A(V)$ *and* $\mathrm{Soc}_A(V) = A\,\mathrm{Soc}_{eAe}(eV)$.

Proof. By the help of the results in [5, pp. 83–87], it remains to prove that $AT = AeT$ is a simple A-submodule of V for any simple eAe-submodule T of eV. In general AT may not be simple, but our hypothesis on V forces it to be simple because any nonzero A-submodule U of AT is not annihilated by e so that $eU = T$ implying $U = AT$. $\square$

Let S and V be modules of an algebra A where S is simple and V is finite dimensional. By the multiplicity of S in V we mean the number of composition factors of V isomorphic to S.

Theorem 3.4. *Let $H \leqslant K \leqslant G$ and let W be a simple $\mathbb{K}\overline{N}_K(H)$-module. Let*

$$M = \uparrow_K^G S_{H,W}^K \quad \text{and} \quad V = \uparrow_{\overline{N}_K(H)}^{\overline{N}_G(H)} W.$$

Then, there is a bijective correspondence between the simple $\mu_{\mathbb{K}}(G)$-submodules of M and the simple $\mathbb{K}\overline{N}_G(H)$-submodules of V. More precisely, any simple $\mu_{\mathbb{K}}(G)$-submodule of M is isomorphic to a simple functor of the form $S_{H,U}^G$ where U is a simple $\mathbb{K}\overline{N}_G(H)$-submodule of V, and conversely any simple $\mathbb{K}\overline{N}_G(H)$-submodule of V is isomorphic to a simple module of the form $S(H)$ where S is a simple $\mu_{\mathbb{K}}(G)$-submodules of M. Furthermore, for any simple $\mathbb{K}\overline{N}_G(H)$-module U, the multiplicity of $S_{H,U}^G$ in $\mathrm{Soc}(M)$ *is equal to the multiplicity of U in* $\mathrm{Soc}(V)$.

Proof. Let $A = \mu_{\mathbb{K}}(G)$, $B = \mathbb{K}\overline{N}_G(H)$ and $e = t_H^H$. By 2.4 the B-modules $eM = M(H)$ and V are isomorphic. We also see by using 3.2 that the ideal I_H of $eAe = A_H \oplus I_H$ given in 2.4 annihilates eM where the algebra A_H is isomorphic to B via $c_H^g \leftrightarrow gH$. Therefore, the (simple) eAe-submodules of eM and the (simple) B-submodules of eM coincide. 3.2 implies that any nonzero A-submodule of M has H as a minimal subgroup. In particular, M has no nonzero A-submodule annihilated by e so that 3.3 may be applied to deduce that there is a bijection between the simple A-submodules of M and the simple B-submodules of $eM \cong V$. Moreover, the B-modules $e\,\mathrm{Soc}(M)$ and $\mathrm{Soc}(V)$ are isomorphic.

Any simple subfunctor S of M has H as a minimal subgroup (by 3.2), and by part (1) of 3.3 the B-module $eS = S(H)$ is a simple B-submodule of $eM \cong V$. So, any simple A-submodule of M is isomorphic to a simple functor of the form $S^G_{H,U}$ where U is a simple B-submodule of V. Conversely, if U is a simple B-submodule of $V \cong eM$ then again by part (1) of 3.3 there is a simple A-submodule S of M such that $S(H) \cong U$.

Let U be a simple B-module. $e\,\mathrm{Soc}(M)$ and $\mathrm{Soc}(V)$ are isomorphic B-modules and any simple A-submodule of M is of the form $S^G_{H,U'}$. By 2.2 we see that the isomorphisms of the simple functors of the forms $S^G_{H,U'}$ and $S^G_{H,U''}$ is equivalent to the isomorphisms of the simple B-modules U' and U''. Therefore, the statement about the multiplicities must be true because $S^G_{H,U'}(H) \cong U'$ and because the left multiplication by the idempotent e respects the direct sums. $\square$

Lemma 3.5. *Let K be a subgroup of G. Then*:

(1) *Let $\mathcal{X}$ be a set of subgroups of K and let T be a $\mu_{\mathbb{K}}(K)$-module. If T is generated as a $\mu_{\mathbb{K}}(K)$-module by its values on $\mathcal{X}$, then $\uparrow^G_K T$ is generated as a $\mu_{\mathbb{K}}(G)$-module by its values on $\mathcal{X}$. In particular, for any simple $\mu_{\mathbb{K}}(K)$-module $S^K_{H,W}$ and any proper $\mu_{\mathbb{K}}(G)$-submodule M of $\uparrow^G_K S^K_{H,W}$, the minimal subgroups of $(\uparrow^G_K S^K_{H,W})/M$ are precisely the G-conjugates of H.*

(2) *Let $\mathcal{Y}$ be a set of subgroups of G and let M be a $\mu_{\mathbb{K}}(G)$-module. If M is generated as a $\mu_{\mathbb{K}}(G)$-module by its values on $\mathcal{Y}$, then $\downarrow^G_K M$ is generated as a $\mu_{\mathbb{K}}(K)$-module by its values on the elements of the set $\{X \leqslant K\colon X \leqslant_G Y,\ Y \in \mathcal{Y}\}$. In particular, for any simple $\mu_{\mathbb{K}}(G)$-module $S^G_{L,V}$ with $L \leqslant_G K$ and any proper $\mu_{\mathbb{K}}(K)$-submodule T of $\downarrow^G_K S^G_{L,V}$, there is a minimal subgroup of $(\downarrow^G_K S^G_{L,V})/T$ which is a G-conjugate of L.*

Proof. (1) Let S be a $\mu_{\mathbb{K}}(G)$-submodule of $\uparrow^G_K T$ such that $S(X) = (\uparrow^G_K T)(X)$ for all X in $\mathcal{X}$. To show that $\uparrow^G_K T$ is generated by its values on $\mathcal{X}$ it suffices to show that $S = \uparrow^G_K T$.

If S is not equal to $\uparrow^G_K T$ then by the adjointness of the pair $(\uparrow^G_K, \downarrow^G_K)$ there is a nonzero $\mu_{\mathbb{K}}(K)$-module homomorphism $\pi : T \to \downarrow^G_K ((\uparrow^G_K T)/S)$ whose L-component

$$\pi_L : T(L) \to \downarrow^G_K \big(\big(\uparrow^G_K T\big)/S\big)(L)$$

is nonzero for some subgroup L of K. So there is a $t \in T(L)$ such that $\pi_L(t) \neq 0$. As T is generated by its values on $\mathcal{X}$,

$$T(L) = \sum_{X \in \mathcal{X}} t^L_L \mu_{\mathbb{K}}(K) t^X_X T$$

so that t can be written as a sum of elements of the form $t^L_{{}^kJ} c^k_J r^X_J t_X$ where $k \in K$, $J \leqslant K$, and $t_X \in T(X)$. Since π commutes with the maps t, r, c of T, it follows that $\pi_L(t)$ can be written as a sum of elements of the form $t^L_{{}^kJ} c^k_J r^X_J \pi_X(t_X)$. But then $\pi_X(t_X)$ and hence $\pi_L(t)$ is 0 because $S(X) = (\uparrow^G_K T)(X)$. Consequently, $S = \uparrow^G_K T$.

For the second statement, let M be a proper $\mu_{\mathbb{K}}(G)$-submodule of $\uparrow^G_K S^K_{H,W}$. As $S^K_{H,V}$ is generated by its value on H, it follows by what we have showed above that the quotient $(\uparrow^G_K S^K_{H,W})/M$ is nonzero at H. Moreover, if Y is a minimal subgroup of the quotient then $\uparrow^G_K S^K_{H,W}$ is nonzero at Y so that $H \leqslant_G Y$ by part (1) of 3.2. Hence, the minimal subgroups of the quotient are precisely the G-conjugates of H.

(2) The first statement is obvious. For the second statement, let T be a proper $\mu_{\mathbb{K}}(K)$-submodule of $\downarrow^G_K S^G_{L,V}$. If the quotient $(\downarrow^G_K S^G_{L,V})/T$ is nonzero at a subgroup X of K then $\downarrow^G_K S^G_{L,V}$ is nonzero at X so that $L \leqslant_G X$. On the other hand, $\downarrow^G_K S^G_{L,V}$ is generated by its values on G-conjugates of L that are in K and so, by the first statement, the quotient cannot be 0 at every G-conjugate of L that is in K. Consequently, a minimal subgroup of the quotient must be a G-conjugate of L. $\square$

Theorem 3.6. *Let $H \leqslant K \leqslant G$ and let W be a simple $\mathbb{K}\overline{N}_K(H)$-module. Then, $\uparrow_K^G S_{H,W}^K$ is a simple (respectively, semisimple) $\mu_{\mathbb{K}}(G)$-module if and only if $\uparrow_{\overline{N}_K(H)}^{\overline{N}_G(H)} W$ is a simple (respectively, semisimple) $\mathbb{K}\overline{N}_G(H)$-module.*

Proof. Let $M = \uparrow_K^G S_{H,W}^K$, $V = \uparrow_{\overline{N}_K(H)}^{\overline{N}_G(H)} W$, $A = \mu_{\mathbb{K}}(G)$, and $B = \mathbb{K}\overline{N}_G(H)$. It follows by 2.4 that $M(H) \cong V$ as B-modules. We note also that the ideal I_H in 2.4 annihilates $M(H)$ which is a consequence of 3.2.

Suppose that M is a simple (respectively, semisimple) A-module. Then 3.2, 3.3 and 2.4 imply that $M(H)$ is a simple (respectively, semisimple) A_H-module. Since A_H and B are isomorphic algebras via $c_H^g \mapsto gH$, we can conclude that V is a simple (respectively, semisimple) B-module.

Suppose that V is a simple (respectively, semisimple) B-module. Then 2.4 implies that $M(H)$ is a simple (respectively, semisimple) eAe-module where $e = t_H^H$. From 3.3 we see that $\mathrm{Soc}_A(M) = AM(H)$ is a simple (respectively, semisimple) A-module. As $S_{H,W}^K$ is generated as a $\mu_{\mathbb{K}}(K)$-module by its value on H, it follows by 3.5 that M is generated as an A-module by $M(H)$. This shows that $M = AM(H) = \mathrm{Soc}_A(M)$. $\square$

The previous result generalizes [13, Proposition 3.5 and Corollary 3.7].

Let e be an idempotent of an algebra A, and let V be an A-module, and T be an eAe-submodule of eV. We denote by the notation $(V :_e T)$ the subset $\{v \in V\colon\ eAv \subseteq T\}$ of V. It is clear that $(V :_e T)$ is an A-submodule of V such that $e(V :_e T) = T$.

Lemma 3.7. *Let A be a finite dimensional $\mathbb{K}$-algebra and let e be a nonzero idempotent of A. If V is a nonzero A-module having no nonzero quotient module annihilated by e (equivalently, $AeV = V$) then:*

(1) *The maps $J \to eJ$ and $(V :_e I) \leftarrow I$ define a bijective correspondence between the maximal A-submodules of V and the maximal eAe-submodules of eV.*
(2) $\mathrm{Jac}_{eAe}(eV) = e\,\mathrm{Jac}_A(V)$ *and* $\mathrm{Jac}_A(V) = (V :_e \mathrm{Jac}_{eAe}(eV))$.

Proof. (1) For any maximal eAe-submodule I of eV, we must show that $(V :_e I)$ is a maximal A-submodule of V and that $e(V :_e I) = I$:

The equality $e(V :_e I) = I$ is clear. It follows from $e(V :_e I) = I$ that $(V :_e I)$ is a proper A-submodule of V. Let T be a proper A-submodule of V containing $(V :_e I)$. Then $I \subseteq eT$. Moreover, V/T, being nonzero, is not annihilated by e so that $eT \neq eV$. Now $I = eT$ by the maximality of I. This implies that $T \subseteq (V :_e I)$. Consequently, $(V :_e I)$ is a maximal A-submodule of V.

For any maximal A-submodule J of V, we must show that eJ is a maximal eAe-submodule of eV and that $(V :_e eJ) = J$:

As V/J is a simple A-module not annihilated by e, the eAe-module $eV/eJ \cong e(V/J)$ is simple so that eJ is a maximal eAe-submodule of eV.

The containment $J \subseteq (V :_e eJ)$ is clear. If $(V :_e eJ)$ is equal to V then $eJ = e(V :_e eJ) = eV$ which is not the case. Hence $(V :_e eJ) = J$ by the maximality of J.

(2) This is obvious from the first part. $\square$

Theorem 3.8. *Let $H \leqslant K \leqslant G$ and let W be a simple $\mathbb{K}\overline{N}_K(H)$-module. Let*

$$M = \uparrow_K^G S_{H,W}^K \quad \text{and} \quad V = \uparrow_{\overline{N}_K(H)}^{\overline{N}_G(H)} W.$$

Then, there is a bijective correspondence between the maximal $\mu_{\mathbb{K}}(G)$-submodules of M and the maximal $\mathbb{K}\overline{N}_G(H)$-submodules of V. In particular, any simple quotient of M is isomorphic to a simple functor of the form $S_{H,U}^G$ where U is a simple quotient of V, and conversely any simple quotient of V is isomorphic to a simple module of the form $S(H)$ where S is a simple quotient of M. Furthermore, for any simple $\mathbb{K}\overline{N}_G(H)$-module U, the multiplicity of $S_{H,U}^G$ in $M/\mathrm{Jac}(M)$ is equal to the multiplicity of U in $V/\mathrm{Jac}(V)$.

Proof. Using 3.5 and 3.7, it can be proved by arguing as in the proof of 3.4. □

Lemma 3.9. *Let A be a finite dimensional $\mathbb{K}$-algebra and e be a nonzero idempotent of A. Suppose that V and W be nonzero A-modules. Let*

$$\phi : \mathrm{Hom}_A(V, W) \to \mathrm{Hom}_{eAe}(eV, eW), \qquad f \mapsto f|_{eV},$$

be the $\mathbb{K}$-space ($\mathbb{K}$-algebra if $W = V$) homomorphism sending f to $f|_{eV}$ where $f|_{eV}$ denotes the restriction of f to eV. Then:

(1) *ϕ is a monomorphism if and only if W has no nonzero A-submodule annihilated by e and isomorphic to a quotient of V.*
(2) *If V has no nonzero quotient module annihilated by e (equivalently, $AeV = V$) and if W has no nonzero A-submodule annihilated by e (equivalently, $(W :_e 0) = 0$), then ϕ is an isomorphism.*

Proof. (1) Firstly, it is obvious that ϕ is not injective if and only if $ef(V) = 0$ for some nonzero f in $\mathrm{Hom}_A(V, W)$. For any A-submodule W_0 of W isomorphic to a quotient V/V_0 of V, it is clear that there is an f in $\mathrm{Hom}_A(V, W)$ with the kernel equal to V_0 and the image equal to W_0. And conversely, any A-module homomorphism gives such submodules. Thus the result follows.

(2) By the first part, it is enough to show that ϕ is surjective:

Let g be in $\mathrm{Hom}_{eAe}(eV, eW)$. We want to construct an element f in $\mathrm{Hom}_A(V, W)$ whose restriction to eV is equal to g. As $V = AeV$, any element of V can be written as a sum of elements of the form aev where each a in A and each v in V. Letting

$$v = a_1 ev_1 + \cdots + a_n ev_n,$$

it is natural to define

$$f(v) = a_1 g(ev_1) + \cdots + a_n g(ev_n).$$

By its construction, we only need to show that f is well-defined because there may be some elements of V which can be expressed as a sum of elements of the form aev in different ways. Suppose that

$$b_1 eu_1 + \cdots + b_m eu_m = 0$$

for some natural number m and some elements $u_i \in V$ and $b_i \in A$. Then for any a in A we have

$$0 = g(0) = g\big(ea(b_1 eu_1 + \cdots + b_m eu_m)\big) = ea\big(b_1 g(eu_1) + \cdots + b_m g(eu_m)\big).$$

Thus $eAw = 0$ where $w = b_1 g(eu_1) + \cdots + b_m g(eu_m)$, implying that Aw is an A-submodule of W annihilated by e. By the condition on W we must have that $w = 0$, as desired. □

Lemma 3.10. *Let A be a finite dimensional $\mathbb{K}$-algebra and e be a nonzero idempotent of A. Let V be a nonzero A-module satisfying $AeV = V$ and $(V :_e 0) = 0$. Suppose*

$$V = V_1 \oplus \cdots \oplus V_n$$

is a decomposition of V into nonzero A-modules. Then,

$$eV = eV_1 \oplus \cdots \oplus eV_n$$

is a decomposition of eV into nonzero eAe-modules such that the A-modules V_i and V_j are isomorphic if and only if the eAe-modules eV_i and eV_j are isomorphic. Moreover, V_i is an indecomposable A-module if and only if eV_i is an indecomposable eAe-module.

Proof. This is obvious because the endomorphism algebras of V and eV are isomorphic by part (2) of 3.9. $\square$

Using 3.10, one may lift most of the results about induction of simple modules of group algebras to the results about induction of simple Mackey functors. As an example, in part (3) of the next result we want to lift a part of the result [6, Theorem 7] which says that if N is a normal subgroup of G and W is a simple $\mathbb{K}N$-module, then, for any indecomposable direct summand P of $\uparrow_N^G W$, there is a simple $\mathbb{K}G$-module V satisfying $\mathrm{Soc}(P) \cong P/\mathrm{Jac}(P) \cong V$ (where W is necessarily a direct summand of $\downarrow_N^G V$). The first two parts of the following result are slight generalizations of 3.4 and 3.8.

Corollary 3.11. *Let $H \leqslant K$ be subgroups of G and let W be a simple $\mathbb{K}\overline{N}_K(H)$-module. Put $A = \mu_{\mathbb{K}}(G)$ and $e = t_H^H$. Then, for any nonzero $\mu_{\mathbb{K}}(G)$-module M:*

(1) *If M is isomorphic to a $\mu_{\mathbb{K}}(G)$-submodule of $\uparrow_K^G S_{H,W}^K$, then the maps $S \to S(H)$ and $AT \leftarrow T$ define a bijective correspondence between the simple $\mu_{\mathbb{K}}(G)$-submodules of M and the simple $\mathbb{K}\overline{N}_G(H)$-submodules of $M(H)$.*
(2) *If M is isomorphic to a quotient functor of $\uparrow_K^G S_{H,W}^K$, then the maps $J \to J(H)$ and $(M :_e I) \leftarrow I$ define a bijective correspondence between the maximal $\mu_{\mathbb{K}}(G)$-submodules of M and the maximal $\mathbb{K}\overline{N}_G(H)$-submodules of $M(H)$.*
(3) *Suppose that $N_K(H)$ is normal in $N_G(H)$. If M is an indecomposable $\mu_{\mathbb{K}}(G)$-module which is a direct summand of $\uparrow_K^G S_{H,W}^K$, then $\mathrm{Soc}(M)$ and $M/\mathrm{Jac}(M)$ are isomorphic simple functors having H as minimal subgroups.*

Proof. Firstly, in all cases the ideal I_H of eAe given in 2.4 annihilates the eAe-module eM so that the eAe-submodules of M and the eAe/I_H-submodules of M are the same, where from 2.4 we also have that $eAe/I_H \cong \mathbb{K}\overline{N}_G(H)$.

(1) Any A-submodule of M is isomorphic to an A-submodule of $\uparrow_K^G S_{H,W}^K$. So 3.2 implies that M has no nonzero A-submodule annihilated by e. The result follows by 3.3.

(2) Any quotient functor of M is isomorphic to a quotient functor of $\uparrow_K^G S_{H,W}^K$. So 3.5 implies that M has no nonzero quotient module annihilated by e. The result follows by 3.7.

(3) In this case any subfunctor and any quotient functor of M are isomorphic to a subfunctor and a quotient functor of $\uparrow_K^G S_{H,W}^K$, respectively. This means that $AeM = M$ and $(M :_e 0) = 0$ implying applicability of 3.10. Now, 3.10 implies that $M(H)$ is an indecomposable $\mathbb{K}\overline{N}_G(H)$-module which is a direct summand of $(\uparrow_K^G S_{H,W}^K)(H)$, isomorphic by 2.4 to $\uparrow_{\overline{N}_K(H)}^{\overline{N}_G(H)} W$. Then the result [6, Theorem 7], mentioned above, implies that

$$\mathrm{Soc}\big(M(H)\big) \cong M(H)/\mathrm{Jac}\big(M(H)\big) \cong V$$

where V is a simple $\mathbb{K}\overline{N}_G(H)$-module. The bijective correspondences given in the first two parts now imply that $\mathrm{Soc}(M) \cong S_{H,V}^G \cong M/\mathrm{Jac}(M)$. $\square$

Theorem 3.12. *Let $K \leqslant G \geqslant L$ and $H \leqslant K \cap L$. Then, for any simple $\mathbb{K}\overline{N}_K(H)$-module W and any simple $\mathbb{K}\overline{N}_L(H)$-module U,*

$$\mathrm{Hom}_{\mu_{\mathbb{K}}(G)}\big(\uparrow_K^G S_{H,W}^K, \uparrow_L^G S_{H,U}^L\big) \cong \mathrm{Hom}_{\mathbb{K}\overline{N}_G(H)}\big(\uparrow_{\overline{N}_K(H)}^{\overline{N}_G(H)} W, \uparrow_{\overline{N}_L(H)}^{\overline{N}_G(H)} U\big)$$

as $\mathbb{K}$-spaces ($\mathbb{K}$-algebras if $L = K$ and $U = W$).

Proof. Let $M_1 = \uparrow_K^G S_{H,W}^K$, $M_2 = \uparrow_L^G S_{H,U}^L$, $A = \mu_{\mathbb{K}}(G)$, and $e = t_H^H$. It is a consequence of 3.2 and 3.5 that both of the modules M_1 and M_2 have no nonzero quotient modules annihilated by e and no nonzero submodules annihilated by e. Thus part (2) of 3.9 implies that $\mathrm{Hom}_A(M_1, M_2)$ and $\mathrm{Hom}_{eAe}(eM_1, eM_2)$ are isomorphic. Moreover, as the ideal I_H of eAe in 2.4 annihilates both of the eAe-modules eM_1 and eM_2, it follows that $\mathrm{Hom}_{eAe}(eM_1, eM_2)$ and $\mathrm{Hom}_{eAe/I_H}(eM_1, eM_2)$ are isomorphic. The result follows from 2.4. $\square$

The previous theorem can also be proved directly by using 2.7 and using the Mackey decomposition formula (for Mackey functors and for modules over group algebras).

For $L = K = G$, the previous theorem reduces to [1, Lemma 11.6.6, p. 302] proved (more conceptually) by using the G-set definition of Mackey functors.

The results 3.4 and 3.8 follows also (more quickly) from the previous theorem.

Let K be a subgroup of G. For a simple $\mu_{\mathbb{K}}(K)$-module $S_{H,W}^K$, an immediate consequence of 3.12 is that $\uparrow_K^G S_{H,W}^K$ is an indecomposable $\mu_{\mathbb{K}}(G)$-module if and only if $\uparrow_{\overline{N}_K(H)}^{\overline{N}_G(H)} W$ is an indecomposable $\mathbb{K}\overline{N}_G(H)$-module.

Corollary 3.13. *Let M be a $\mu_{\mathbb{K}}(G)$-module, let H be a subgroup of G, and let U be a simple $\mathbb{K}\overline{N}_G(H)$-module. Then, the multiplicity of $S_{H,U}^G$ in the socle (respectively, in the head) of M is equal to the multiplicity of $S_{H,U}^{N_G(H)}$ in the socle (respectively, in the head) of $\downarrow_{N_G(H)}^G M$.*

Proof. As a consequence of 3.12 the endomorphism algebra of the $\mu_{\mathbb{K}}(G)$-module $S_{H,V}^G$ is isomorphic to the endomorphism algebra of the $\mu_{\mathbb{K}}(N_G(H))$-module $S_{H,V}^{N_G(H)}$. Using the isomorphism $S_{H,V}^G \cong \uparrow_{N_G(H)}^G S_{H,V}^{N_G(H)}$ given in 2.7, we see that the result follows by the adjointness of the pair $(\uparrow_{N_G(H)}^G, \downarrow_{N_G(H)}^G)$ (respectively, of the pair $(\downarrow_{N_G(H)}^G, \uparrow_{N_G(H)}^G)$). $\square$

It may be thought that 3.12 is a very restrictive result dealing with simple functors whose minimal subgroups are equal (or conjugate). Indeed, the next result indicates that it is not so.

Proposition 3.14. *Let $A \leqslant K \leqslant G \geqslant L \geqslant B$. Then, for any simple $\mathbb{K}\overline{N}_K(A)$-module W and any simple $\mathbb{K}\overline{N}_L(B)$-module U, if*

$$\mathrm{Hom}_{\mu_{\mathbb{K}}(G)}\big(\uparrow_K^G S_{A,W}^K, \uparrow_L^G S_{B,U}^L\big) \neq 0,$$

then $B = A^g$ for some $g \in G$ (so that $\uparrow_L^G S_{B,U}^L$ and $\uparrow_{{}^gL}^G S_{A,{}^gU}^{{}^gL}$ are isomorphic).

Proof. Let $M_1 = \uparrow_K^G S_{A,W}^K$ and $M_2 = \uparrow_L^G S_{B,U}^L$. Suppose that $\mathrm{Hom}_{\mu_{\mathbb{K}}(G)}(M_1, M_2) \neq 0$. Then, using the adjointness of the pairs $(\uparrow_K^G, \downarrow_K^G)$ and $(\downarrow_L^G, \uparrow_L^G)$, we see that there are (nonzero) maps

$$S_{A,W}^K \to \downarrow_K^G M_2 \quad \text{and} \quad \downarrow_L^G M_1 \to S_{B,U}^L,$$

which are necessarily a $\mu_{\mathbb{K}}(K)$-module monomorphism and a $\mu_{\mathbb{K}}(L)$-module epimorphism, respectively. From these morphisms of functors we obtain that $M_2(A) \neq 0$ and $M_1(B) \neq 0$. So it follows by 2.3 that $B \leqslant_L L \cap A^x$ and that $A \leqslant_K K \cap B^y$ for some x and y in G. Hence, $B = A^g$ for some $g \in G$. Furthermore, the g conjugate $|_G^g M_2$ of the functor M_2 for G is isomorphic to M_2, and hence M_2 is isomorphic to $|_G^g M_2 \cong \uparrow_{{}^gL}^G S_{{}^gB,{}^gU}^{{}^gL}$. $\square$

One may want to obtain results similar to 3.4, 3.6, 3.8 and 3.12 for restrictions of simple functors. The results similar to 3.4 and 3.8 can be readily given by using 3.12 and using the adjointness property of induction and restriction.

Theorem 3.15. *Let $K \leqslant L \leqslant G$ and let V be a simple $\mathbb{K}\overline{N}_G(K)$-module. Let $M = \downarrow_L^G S_{K,V}^G$. Then, any simple $\mu_{\mathbb{K}}(L)$-submodule of M is isomorphic to a simple functor of the form $S^L_{{}^gK,W}$ where g is an element of G with ${}^gK \leqslant L$ and W is a simple $\mathbb{K}\overline{N}_L({}^gK)$-submodule of ${}^gV$. Conversely, for any element g of G with ${}^gK \leqslant L$, any simple $\mathbb{K}\overline{N}_L({}^gK)$-submodule of ${}^gV$ is isomorphic to a simple module of the form $S({}^gK)$ where S is a simple $\mu_{\mathbb{K}}(L)$-submodule of M. Moreover, for any element g of G with ${}^gK \leqslant L$ and any simple $\mathbb{K}\overline{N}_L({}^gK)$-module of W, the multiplicity of $S^L_{{}^gK,W}$ in $\mathrm{Soc}(M)$ is equal to the multiplicity of U in $\mathrm{Soc}(\downarrow^{\overline{N}_G({}^gK)}_{\overline{N}_L({}^gK)} {}^gV)$.*

Proof. It follows by part (2) of 3.2 that any simple $\mu_{\mathbb{K}}(L)$-submodule of M has a minimal subgroup which is a G-conjugate of K so that it must be of the form $S^L_{{}^gK,W}$ where g is an element of G with ${}^gK \leqslant L$ and W is simple $\mathbb{K}\overline{N}_L({}^gK)$-submodule of ${}^gV \cong M({}^gK)$.

What remains will follow easily from the following isomorphism of $\mathbb{K}$-spaces. Let $g \in G$ with ${}^gK \leqslant L$ and let W be a simple $\mathbb{K}\overline{N}_L({}^gK)$-module. Put $x = g^{-1}$ to simplify the notation. Using the adjointness of the pair $(\uparrow_L^G, \downarrow_L^G)$ and 3.12 we have the following isomorphisms of $\mathbb{K}$-spaces:

$$\begin{aligned}
\mathrm{Hom}_{\mu_{\mathbb{K}}(L)}\big(S^L_{{}^gK,W}, M\big) &\cong \mathrm{Hom}_{\mu_{\mathbb{K}}(G)}\big(\uparrow_L^G S^L_{{}^gK,W}, S^G_{K,V}\big)\\
&\cong \mathrm{Hom}_{\mu_{\mathbb{K}}(G)}\big(\uparrow_L^G |_L^g S^{L^g}_{K,{}^xW}, S^G_{K,V}\big)\\
&\cong \mathrm{Hom}_{\mu_{\mathbb{K}}(G)}\big(|_G^g \uparrow_{L^g}^G S^{L^g}_{K,{}^xW}, S^G_{K,V}\big)\\
&\cong \mathrm{Hom}_{\mu_{\mathbb{K}}(G)}\big(\uparrow_{L^g}^G S^{L^g}_{K,{}^xW}, S^G_{K,V}\big)\\
&\cong \mathrm{Hom}_{\mathbb{K}\overline{N}_G(K)}\big(\uparrow^{\overline{N}_G(K)}_{\overline{N}_{L^g}(K)} {}^xW, V\big)\\
&\cong \mathrm{Hom}_{\mathbb{K}\overline{N}_{L^g}(K)}\big({}^xW, \downarrow^{\overline{N}_G(K)}_{\overline{N}_{L^g}(K)} V\big)\\
&\cong \mathrm{Hom}_{\mathbb{K}\overline{N}_L({}^gK)}\big(W, \downarrow^{\overline{N}_G({}^gK)}_{\overline{N}_L({}^gK)} {}^gV\big).
\end{aligned}$$

We also used the following obvious properties of conjugation which transports the structure. Firstly, the Mackey functors $S^L_{{}^gK,W}$ and $|_L^g S^{L^g}_{K,{}^xW}$, where $x = g^{-1}$, are isomorphic. Secondly, given subgroups $A \leqslant B \leqslant G$, an element $g \in G$, and $\mathbb{K}A$-modules U_1 and U_2, the functors $|_B^g \uparrow_A^B$ and $\uparrow_{{}^gA}^{{}^gB} |_A^g$ are naturally isomorphic, the $\mathbb{K}$-spaces $\mathrm{Hom}_{\mathbb{K}A}(U_1, U_2)$ and $\mathrm{Hom}_{\mathbb{K}({}^gA)}({}^gU_1, {}^gU_2)$ are isomorphic, and moreover $|_G^g$ and the identity functor are naturally isomorphic. $\square$

The previous theorem remains true if we replace simple $\mu_{\mathbb{K}}(L)$ and $\mathbb{K}\overline{N}_L({}^gK)$-submodules with simple quotients, and replace socles with heads.

Theorem 3.16. *Let $K \leqslant L \leqslant G$ and let V be a simple $\mathbb{K}\overline{N}_G(K)$-module. Let $M = \downarrow_L^G S^G_{K,V}$.*

(1) *M is a semisimple $\mu_{\mathbb{K}}(L)$-module if and only if ${}^gV$ is a semisimple $\overline{N}_L({}^gK)$-module for every element g of G with ${}^gK \leqslant L$.*
(2) *M is a simple $\mu_{\mathbb{K}}(L)$-module if and only if any element of the set $\{{}^gK\colon {}^gK \leqslant L,\ g \in G\}$ is an L-conjugate of K and the $\mathbb{K}\overline{N}_L(K)$-module V is simple.*

Proof. As a consequence of 3.15, for any $g \in G$ with ${}^gK \leqslant L$ we have

$$\big(\mathrm{Soc}(M)\big)\big({}^gK\big) \cong \mathrm{Soc}\big(\downarrow^{\overline{N}_G({}^gK)}_{\overline{N}_L({}^gK)} {}^gV\big).$$

(1) Suppose that M is semisimple. Then $M = \mathrm{Soc}(M)$ so that the socle of the $\overline{N}_L({}^gK)$-module ${}^gV$ is isomorphic to $M({}^gK)$. As $M({}^gK) \cong {}^gV$, the $\overline{N}_L({}^gK)$-module ${}^gV$ must be semisimple. Suppose that ${}^gV$

is semisimple for every g in G with ${}^gK \leqslant L$. Since $M({}^gK) \cong {}^gV$, we must have that $(\mathrm{Soc}(M))({}^gK) = M({}^gK)$. It follows by part (2) of 3.5 that M is generated by its values on ${}^gK$ where g ranges over elements of G satisfying ${}^gK \leqslant L$. This shows that $M = \mathrm{Soc}(M)$.

(2) This is clear from 3.15 and from the isomorphism given at the beginning of the proof. □

The following immediate consequence of 3.15 and 3.16 generalizes part (ii) of [14, Corollary 3.5].

Corollary 3.17. *Let $K \leqslant_G L$ be subgroups of G and let V be a simple $\mathbb{K}\overline{N}_G(K)$-module such that $\dim_{\mathbb{K}} V = 1$. Then, the $\mu_{\mathbb{K}}(L)$-module $\downarrow_L^G S_{K,V}^G$ is semisimple and satisfies*

$$\downarrow_L^G S_{K,V}^G \cong \bigoplus_{LgN_G(K) \subseteq G:\ {}^gK \leqslant L} S^L_{{}^gK,{}^gV}.$$

We now want to obtain an analogous of 3.12 for restrictions of simple functors. It seems that such a result is not an immediate consequences of 3.12, the Mackey decomposition formula, the formula in 2.7, and the adjointness properties of restriction and induction. Instead of using 3.12 we may try to adopt the proof of 3.12. Therefore, given a simple functor M for G and a subgroup L of G, we must find an appropriate idempotent e of $\mu_{\mathbb{K}}(L)$ such that $\downarrow_L^G M$ has no nonzero quotient module annihilated by e and no nonzero submodule annihilated by e. We must also relate the algebra $e\mu_{\mathbb{K}}(L)e$ to some group algebras.

Lemma 3.18. *Let $\mathcal{X}$ be a set of subgroups of G and let*

$$e_{\mathcal{X}} = \sum_{X \in \mathcal{X}} t_X^X.$$

Then we have the direct sum decomposition

$$e_{\mathcal{X}} \mu_{\mathbb{K}}(G) e_{\mathcal{X}} = A_{\mathcal{X}} \oplus I_{\mathcal{X}}$$

where $A_{\mathcal{X}}$ is a unital subalgebra of $e_{\mathcal{X}}\mu_{\mathbb{K}}(G)e_{\mathcal{X}}$ and $I_{\mathcal{X}}$ is a $(A_{\mathcal{X}}, A_{\mathcal{X}})$-bisubmodule of $e_{\mathcal{X}}\mu_{\mathbb{K}}(G)e_{\mathcal{X}}$. The elements of the form $t^X_{{}^gJ} c^g_J r^Y_J$ where X and Y are different elements of $\mathcal{X}$ and the elements of the form $t^X_{{}^gJ} c^g_J r^X_J$ where $X \in \mathcal{X}$ and $J \neq X$ form a $\mathbb{K}$-basis of $I_{\mathcal{X}}$. Moreover, we have the following $\mathbb{K}$-algebra isomorphism

$$A_{\mathcal{X}} = \bigoplus_{X \in \mathcal{X}} A_X, \quad A_X = \left(\bigoplus_{gX \subseteq N_G(X)} \mathbb{K}c_X^g\right) \cong \mathbb{K}\overline{N}_G(X),\ c_X^g \leftrightarrow gX,$$

where A_X are two sided ideals of $A_{\mathcal{X}}$ so that the identities $t_X^X = c_X^1$ of the algebras A_X, $X \in \mathcal{X}$, are mutually orthogonal central idempotents of $A_{\mathcal{X}}$ whose sum is equal to the identity $e_{\mathcal{X}}$ of $A_{\mathcal{X}}$. Furthermore, $I_{\mathcal{X}}$ is a two sided ideal of $e_{\mathcal{X}}\mu_{\mathbb{K}}(G)e_{\mathcal{X}}$ if and only if there are no elements X and Y of $\mathcal{X}$ with $X < Y$.

Proof. Follows easily by the axioms defining the Mackey algebra and by the basis Theorem 2.1. See also 2.4. □

Using the previous result and 3.9, we sometimes can reduce hom spaces of Mackey functors to hom spaces of $A_{\mathcal{X}}$-modules. Moreover, as the algebra direct summands of $A_{\mathcal{X}}$ given in 3.18 are actually two sided ideals of $A_{\mathcal{X}}$, using the next result, hom spaces can be reduced further to direct sums of hom spaces of group algebras.

Remark 3.19. Let $1 = e_1 + \cdots + e_n$ be a decomposition of the unity of a finite dimensional $\mathbb{K}$-algebra A into orthogonal central idempotents. Then, for any A-modules V and W,

$$\mathrm{Hom}_A(V, W) \to \bigoplus_{i=1}^{n} \mathrm{Hom}_{Ae_i}(e_i V, e_i W), \qquad f \mapsto \bigoplus_{i=1}^{n} f|_{e_i V},$$

is a $\mathbb{K}$-space ($\mathbb{K}$-algebra if $V = W$) isomorphism.

Proof. Well-known and easy. $\square$

Theorem 3.20. *Let $K \leqslant L \leqslant A \leqslant G \geqslant B \geqslant L$. Let $Y_1, Y_2, \ldots, Y_n$ be a complete list of representatives of L-orbits (i.e., L-conjugacy classes) on the L-set*

$$\left\{{}^aK\colon {}^aK \leqslant L,\ a \in A\right\} \cap \left\{{}^bK\colon {}^bK \leqslant L,\ b \in B\right\}$$

on which L acts by conjugation. Suppose that

$$Y_i = {}^{a_i}K = {}^{b_i}K; \quad a_i \in A,\ b_i \in B,\ i = 1, 2, \ldots, n.$$

Then, for any simple $\mathbb{K}\overline{N}_A(K)$-module W and any simple $\mathbb{K}\overline{N}_B(K)$-module U,

$$\mathrm{Hom}_{\mu_{\mathbb{K}}(L)}\left(\downarrow_L^A S_{K,W}^A, \downarrow_L^B S_{K,U}^B\right) \cong \bigoplus_{i=1}^{n} \mathrm{Hom}_{\mathbb{K}\overline{N}_L(Y_i)}\left({}^{a_i}W, {}^{b_i}U\right)$$

as $\mathbb{K}$-spaces ($\mathbb{K}$-algebras if $B = A$ and $W = U$ and if we choose $b_i = a_i$).

Proof. Let $X_1, X_2, \ldots, X_m$ be a complete list of representatives of L-orbits (i.e., L-conjugacy classes) on the L-set

$$\left\{{}^aK\colon {}^aK \leqslant L,\ a \in A\right\} \cup \left\{{}^bK\colon {}^bK \leqslant L,\ b \in B\right\}$$

on which L acts by conjugation. We may assume that $\{Y_1, Y_2, \ldots, Y_n\} \subseteq \{X_1, X_2, \ldots, X_m\}$.

We also let $M_1 = \downarrow_L^A S_{K,W}^A$, $M_2 = \downarrow_L^B S_{K,U}^B$, $E = \mu_{\mathbb{K}}(L)$, and $\mathcal{X} = \{X_1, X_2, \ldots, X_m\}$.

Letting $e_{\mathcal{X}}$ be the idempotent of E defined as in 3.18, it follows by part (2) of 3.5 that the E-module M_1 has no nonzero quotient module annihilated by $e_{\mathcal{X}}$, because any quotient of M_1 must be nonzero at some element of $\mathcal{X}$. And similarly, it follows by part (2) of 3.2 that M_2 has no nonzero E-submodule annihilated by $e_{\mathcal{X}}$. Then 3.9 implies that

$$\mathrm{Hom}_E(M_1, M_1) \cong \mathrm{Hom}_{e_{\mathcal{X}}Ee_{\mathcal{X}}}(e_{\mathcal{X}}M_1, e_{\mathcal{X}}M_2).$$

If $M_1(J) \neq 0$ for some subgroup J of L then $S_{K,W}^A(J) \neq 0$ implying that $K \leqslant_A J$. This shows that the ideal $I_{\mathcal{X}}$ of $e_{\mathcal{X}}Ee_{\mathcal{X}}$ given in 3.18 annihilates the $e_{\mathcal{X}}Ee_{\mathcal{X}}$-module $e_{\mathcal{X}}M_1$, because if a basis element $t_{{}^gJ}^X c_J^g r_J^Y$ of $I_{\mathcal{X}}$ does not annihilate $e_{\mathcal{X}}M_1$ then, as $g \in L$, we must have that $X =_L Y$, which is not the case by the choice of the set $\mathcal{X}$. In a similar way, we see also that $I_{\mathcal{X}}$ annihilates $e_{\mathcal{X}}M_2$. Therefore,

$$\mathrm{Hom}_{e_{\mathcal{X}}Ee_{\mathcal{X}}}(e_{\mathcal{X}}M_1, e_{\mathcal{X}}M_2) \cong \mathrm{Hom}_{A_{\mathcal{X}}}(e_{\mathcal{X}}M_1, e_{\mathcal{X}}M_2)$$

where $A_{\mathcal{X}}$ is the subalgebra of $e_{\mathcal{X}}Ee_{\mathcal{X}}$ given in 3.18.

The unities $t_X^X = c_X^1$ of the algebras A_X are central idempotents of $A_{\mathcal{X}}$ which are mutually orthogonal. That is,

$$\sum_{X \in \mathcal{X}} t_X^X = e_{\mathcal{X}}$$

is a decomposition of the unity $e_{\mathcal{X}}$ of the algebra $A_{\mathcal{X}}$ into central orthogonal idempotents of $A_{\mathcal{X}}$. Now it follows by 3.19 that

$$\mathrm{Hom}_{A_{\mathcal{X}}}(e_{\mathcal{X}} M_1, e_{\mathcal{X}} M_2) \cong \bigoplus_{X \in \mathcal{X}} \mathrm{Hom}_{A_{\mathcal{X}} t_X^X}\bigl(M_1(X), M_2(X)\bigr).$$

Let $X \in \mathcal{X}$. Then $X = {}^gK$ for some $g \in A \cup B$ with ${}^gK \leqslant L$. If

$$\mathrm{Hom}_{A_{\mathcal{X}} t_X^X}\bigl(M_1(X), M_2(X)\bigr) \neq 0$$

then both of $M_1(X)$ and $M_2(X)$ must be nonzero. Thus $S_{K,W}^A({}^gK) \neq 0$ and $S_{K,U}^B({}^gK) \neq 0$. This gives that ${}^gK =_A K$ and ${}^gK =_B K$. Consequently, X must be an L-conjugate of Y_i for some $i \in \{1, 2, \dots, n\}$. Thus,

$$\bigoplus_{X \in \mathcal{X}} \mathrm{Hom}_{A_{\mathcal{X}} t_X^X}\bigl(M_1(X), M_2(X)\bigr) \cong \bigoplus_{i=1}^{n} \mathrm{Hom}_{A_{\mathcal{X}} t_{Y_i}^{Y_i}}\bigl(M_1(Y_i), M_2(Y_i)\bigr).$$

The algebras $A_{\mathcal{X}} t_{Y_i}^{Y_i} = A_{Y_i}$ and $\mathbb{K}\overline{N}_L(Y_i)$ are isomorphic via $c_{Y_i}^{y_i} \mapsto y_i Y_i$ by 3.18. Moreover,

$$M_1(Y_i) = M_1\bigl({}^{a_i}K\bigr) \cong {}^{a_i}W \quad \text{and} \quad M_2(Y_i) = M_2\bigl({}^{b_i}K\bigr) \cong {}^{b_i}U.$$

As a result, for each $i \in \{1, 2, \dots, n\}$, we have that

$$\mathrm{Hom}_{A_{\mathcal{X}} t_{Y_i}^{Y_i}}\bigl(M_1(Y_i), M_2(Y_i)\bigr) \cong \mathrm{Hom}_{\mathbb{K}\overline{N}_L(Y_i)}\bigl({}^{a_i}W, {}^{b_i}U\bigr). \qquad \square$$

Let $H \leqslant K$ be subgroups of G. For a simple $\mu_{\mathbb{K}}(G)$-module $S_{H,V}^G$, it follows easily by 3.20 that the $\mu_{\mathbb{K}}(K)$-module $\downarrow_K^G S_{H,V}^G$ is indecomposable if and only if any element of the set $\{{}^gH\colon {}^gH \leqslant K,\ g \in G\}$ is a K-conjugate of H and the $\mathbb{K}\overline{N}_K(H)$-module $\downarrow_{\overline{N}_K(H)}^{\overline{N}_G(H)} V$ is indecomposable.

Proposition 3.21. *Let $L \leqslant A \leqslant G \geqslant B \geqslant L$ and $X \leqslant A$ and $Y \leqslant B$. Then, for any simple $\mathbb{K}\overline{N}_A(X)$-module W and any simple $\mathbb{K}\overline{N}_Y(K)$-module U, if*

$$\mathrm{Hom}_{\mu_{\mathbb{K}}(L)}\bigl(\downarrow_L^A S_{X,W}^A, \downarrow_L^B S_{Y,U}^B\bigr) \neq 0$$

then $X^a = Y^b \leqslant L$ for some $a \in A$ and $b \in B$.

Proof. Similar to the proof of 3.14. $\square$

As a consequence of 3.21, hom spaces of restrictions of any simple functors can be related to hom spaces of modules of some group algebras.

Given any simple $\mathbb{K}\overline{N}_K(H)$-module W, we have seen that the socle and head of the $\mu_{\mathbb{K}}(G)$-module $M = \uparrow_K^G S_{H,W}^K$ can be determined by the socle and head of the $\mathbb{K}\overline{N}_G(H)$-module $V = \uparrow_{\overline{N}_K(H)}^{\overline{N}_G(H)} W$. As M may have composition factors with minimal subgroups not G-conjugates of H, we do not expect a connection between, say, the socle series of M and V (except when the socle length of M is 2). For instance, letting G be a 2-group, $|G : K| = 2$, $H = K$, and $W = \mathbb{K}$ (the trivial module), if $\mathbb{K}$ is of characteristic 2 then it can be seen that the factors of the socle series of (the uniserial $\mu_{\mathbb{K}}(G)$-module) M are $S^G_{K,\mathbb{K}}$, $S^G_{G,\mathbb{K}}$, and $S^G_{K,\mathbb{K}}$, while $\mathbb{K}$ and $\mathbb{K}$ are the factors of the socle series of V.

4. Brauer quotients and maximal subfunctors

Let M be a $\mu_{\mathbb{K}}(G)$-module and H be a subgroup of G. The purpose of this section is to find some relations between the maximal $\mu_{\mathbb{K}}(G)$-submodules of M and the $\mathbb{K}\overline{N}_G(H)$-submodules of the coordinate module $M(H)$ of M.

We begin with recording some properties of the submodules $(V :_e T)$ defined at the beginning of 3.7

Lemma 4.1. *Let A be a finite dimensional $\mathbb{K}$-algebra and e be a nonzero idempotent of A. Suppose that $W \subseteq V$ are A-modules with $eV \neq 0$ and suppose that I, I_1, and I_2 are eAe-submodules of eV. Then*:

(1) *$(V :_e I)$ is the largest A-submodule of V subject to the condition $e(V :_e I) = I$. In particular, $W \subseteq (V :_e eW)$.*
(2) *If $I_1 \subseteq I_2$ then $(V :_e I_1) \subseteq (V :_e I_2)$.*
(3) *$(V :_e I_1) \cap (V :_e I_2) = (V :_e I_1 \cap I_2)$.*
(4) *$(V :_e I) \cap W = (W :_e I \cap eW)$.*
(5) *$(V/W :_e (I + W)/W) = (V :_e I + eW)/W$.*
(6) *$(V \times V' :_e I \times I') = (V :_e I) \times (V' :_e I')$ for any A-module V' and any eAe-submodule I' of eV'.*
(7) *V is a simple A-module if and only if $AeV = V$, $(V :_e 0) = 0$, and eV is a simple eAe-module.*
(8) *Let $(V :_e 0) = 0$. Then, V is a semisimple A-module if and only if $AeV = V$ and eV is a semisimple eAe-module.*

Proof. Parts (1)–(6) are immediate from the definition of $(V :_e T)$.

(7) Suppose that V is a simple A-module. As eV is nonzero, the A-submodules AeV and $(V :_e 0)$ of V are nonzero and proper, respectively. Thus $AeV = V$ and $(V :_e 0) = 0$. The simplicity of the eAe-module eV is well known (from [5, pp. 83–87] or 3.3). Conversely, suppose V is an A-module satisfying $AeV = V$, $(V :_e 0) = 0$, and eV simple. Let U be a nonzero A-submodule of V. Then it follows from $(V :_e 0) = 0$ that eU is a nonzero eAe-submodule of eV so that $eU = eV$ by the simplicity of eV. Now $AeV = V$ implies that $V = U$. Hence V is a simple A-module.

(8) As $(V :_e 0) = 0$, the A-module V has no nonzero A-submodule annihilated by e so that 3.3 may be applied to see the result. $\square$

Let $\mathcal{X}$ be a set of subgroups of G and M be a $\mu_{\mathbb{K}}(G)$-module. If we put $A = \mu_{\mathbb{K}}(G)$ and $e = e_{\mathcal{X}}$ where the idempotent $e_{\mathcal{X}}$ is defined as in 3.18, then the module $(M :_e 0)$ becomes an already familiar subfunctor of M. Indeed, assuming that $\mathcal{X}$ is closed under taking subgroups and taking G-conjugates, we have

$$\begin{aligned}(M :_e 0) &= \Big\{m = \bigoplus_{H \leqslant G} m_H \in M\colon\ eAm = 0\Big\}\\ &= \bigoplus_{H \leqslant G} \big\{m_H \in M(H)\colon\ t^X_X \mu_{\mathbb{K}}(G) t^H_H m_H = 0,\ \forall X \in \mathcal{X}\big\}.\end{aligned}$$

The basis Theorem 2.1 and the conditions on $\mathcal{X}$ imply that $t^X_X \mu_{\mathbb{K}}(G) t^H_H m_H = 0$ for all $X \in \mathcal{X}$ if and only if $r^H_X m_H = 0$ for all $X \in \mathcal{X}$ satisfying $X \leqslant H$. Consequently,

$$(M :_e 0)(H) = \bigcap_{X \in \mathcal{X}\colon\ X \leqslant H} \operatorname{Ker}\big(r^H_X : M(H) \to M(X)\big).$$

Thus $(M :_e 0)$ is the subfunctor $\operatorname{Ker} r^M_{\mathcal{X}}$ of M defined in [10, Section 3]. This observation shows that part (7) of 4.1 implies the characterization of simple functors in [10, (3.1) Theorem].

Moreover, for any set $\mathcal{X}$ of subgroups of G and any $\mu_{\mathbb{K}}(G)$-module M, a $\mu_{\mathbb{K}}(G)$-submodule $R_{\mathcal{X}}M$ of M defined in [12] to be the largest $\mu_{\mathbb{K}}(G)$-submodule of M subject to the condition $r^K_J(R_{\mathcal{X}}M(K)) = 0$ for all $J \in X$ with $J \leqslant K$. It can be seen easily that $R_{\mathcal{X}}M = (M :_{e_{\mathcal{X}}} 0)$.

For an algebra A and its idempotent e we want to relate the maximal A-submodules of an A-module V to the maximal eAe-submodules of eV. Although we gave such a relation in 3.7, some modules we want to consider may not satisfy the conditions of 3.7. For this reason we next state the following result.

Lemma 4.2. *Let A be a finite dimensional $\mathbb{K}$-algebra and e be a nonzero idempotent of A. Suppose that V is a nonzero A-module, J is an A-submodule of V, and I is an eAe-submodule of eV. Then*:

(1) *I is a maximal eAe-submodule of eV if and only if $(V :_e I)$ is a largest element of the set of all A-submodules of V not containing AeV.*
(2) *$(V :_e I)$ is a maximal A-submodule of V if and only if I is a maximal eAe-submodule of eV and $AeV + (V :_e I) = V$.*
(3) *J is a largest element of the set of all A-submodules of V not containing AeV if and only if eJ is a maximal eAe-submodule of eV and $J = (V :_e eJ)$.*
(4) *$J = (V :_e eJ)$ if and only if V/J has no nonzero A-submodule annihilated by e, equivalently $(V/J :_e 0) = 0$.*
(5) *Suppose that J does not contain AeV. Then, J is a maximal A-submodule of V if and only if eJ is a maximal eAe-submodule of eV, $AeV + J = V$, and $(V :_e eJ) = J$.*

Proof. (1) Let I be a maximal eAe-submodule of eV. As I is not equal to eV, the A-module $(V :_e I)$ cannot contain AeV. Let W be an A-submodule of V containing $(V :_e I)$ but not containing AeV. Then eW is a proper eAe-submodule of eV containing I. This implies that $eW = I$ because I is a maximal eAe-submodule of eV. Hence $W = (V :_e I)$.

Let $(V :_e I)$ be a largest among all the A-submodules of V not containing AeV. Then I must be a proper eAe-submodule of eV. Let T be a maximal eAe-submodule of eV that contains I. By using 4.1 we see that $(V :_e T)$ contains $(V :_e I)$ but does not contain AeV. Because of the condition on $(V :_e I)$, this implies that $(V :_e T) = (V :_e I)$. Thus $T = I$.

(2) We may assume that I is not equal to eV, because $(V :_e I) = V$ if and only if $I = eV$. Thus, $V/(V :_e I)$ is not annihilated by e so that part (7) of 4.1 is applicable.

$(V :_e I)$ is a maximal A-submodule of V if and only if $V/(V :_e I)$ is a simple A-module. This is equivalent to the conditions: $Ae(V/(V :_e I)) = V/(V :_e I)$, the eAe-module $e(V/(V :_e I))$ is simple, and $(V/(V :_e I) :_e 0) = 0$. The result follows by 4.1.

(3) Let J be such a largest element. As J does not contain AeV, the eAe-module eJ is not equal eV. Let I' be a maximal eAe-submodule of eV containing eJ. It follows by part (1) that the A-module $(V :_e I')$ is also a largest element of the set of all A-submodules of V not containing AeV. This shows that $(V :_e I') = J$ because $(V :_e I')$ contains J. Hence $I' = eJ$ is a maximal eAe-submodule of eV and $J = (V :_e eJ)$. The converse direction follows from the first part of this lemma.

(4) Follows from part (5) of 4.1 which implies that $(V :_e eJ)/J = (V/J :_e 0)$.

(5) Follows from part (7) of 4.1 because the maximality of J is equivalent to simplicity of V/J, which is not annihilated by e. $\square$

From 4.2 the following is immediate.

Proposition 4.3. *Let A be a finite dimensional $\mathbb{K}$-algebra and e be a nonzero idempotent of A. Suppose that V is a nonzero A-module. Then*:

(1) *The maps $J \to eJ$ and $(V :_e I) \leftarrow I$ define a bijective correspondence between the largest elements of the set of all A-submodules of V not containing AeV and the maximal eAe-submodules of eV.*

(2) *The maps $J \to eJ$ and $(V :_e I) \leftarrow I$ define a bijective correspondence between the maximal A-submodules of V that are containing $A(1-e)V$ (so, necessarily not containing AeV) and the maximal eAe-submodules of eV that are containing $eA(1-e)V$.*
(3) *The maps $J \to eJ$ and $(V :_e I) \leftarrow I$ define a bijective correspondence between the maximal A-submodules of V that are not containing AeV and the maximal eAe-submodules of eV that satisfy $AeV + (V :_e I) = V$.*

We next need to recall the notion of the Brauer quotient of a Mackey functor, see [9,11,12]. Let M be a $\mu_{\mathbb{K}}(G)$-module and H be a subgroup of G. We put

$$b_H(M) = \sum_{S<H} t_S^H\big(M(S)\big).$$

It is clear that $b_H(M)$ is a $\mathbb{K}\overline{N}_G(H)$-submodule of $M(H)$. The quotient module $M/b_H(M)$ is called the Brauer quotient (or the residue module) of $M(H)$ and denoted by $\overline{M}(H)$.

Given any $\mu_{\mathbb{K}}(G)$-module M and any subgroup H of G we will observe in the proof of the next result that if I is a (maximal) $\mathbb{K}\overline{N}_G(H)$-submodule of $M(H)$ containing $b_H(M)$ then it is also a $t_H^H\mu_{\mathbb{K}}(G)t_H^H$-submodule of $M(H)$ so that the notation $(M :_e I)$ in the next result makes sense (see also part (1) of 4.5).

Theorem 4.4. *Let M be a $\mu_{\mathbb{K}}(G)$-module and H be a subgroup of G. Put $e = t_H^H$. Then, the maps $J \to J(H)$ and $(M :_e I) \leftarrow I$ define a bijective correspondence between the largest elements of the set of all subfunctors J of M whose quotient functor M/J has H as a minimal subgroup and the maximal $\mathbb{K}\overline{N}_G(H)$-submodules $I/b_H(M)$ of $\overline{M}(H)$. In particular, $\overline{M}(H) = 0$ if and only if M has no quotient functor having H as a minimal subgroup.*

Proof. Let $A = \mu_{\mathbb{K}}(G)$, $B = \mathbb{K}\overline{N}_G(H)$, $\mathcal{X} = \{X \leqslant G\colon\ X < H\}$, and let the idempotent $f = e_{\mathcal{X}}$ of A be defined as in 3.18. By 3.18 or part (1) of 2.4 we have the direct sum decomposition $eAe = A_H \oplus I_H$ where the algebra A_H can be identified with B via the isomorphism given by $c_H^g \leftrightarrow gH$.

We also define five sets $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}$ as follows. $\mathcal{A}$ is the set of all subfunctors of M whose quotient has H as a minimal subgroup, $\mathcal{B}$ is the set of all A-submodules of M containing AfM but not containing AeM, $\mathcal{C}$ is the set of all eAe-submodules of eM containing $eAfM$, $\mathcal{D}$ is the set of all B-submodules of eM containing $eAfM$, and $\mathcal{E}$ is the set of all $\mathbb{K}\overline{N}_G(H)$-submodules of $M(H)$ containing $b_H(M)$.

We first show that the sets $\mathcal{A}$ and $\mathcal{B}$ are equal: Let J be a subfunctor of M. Then, H is a minimal subgroup of M/J if and only if $(M/J)(X) = 0$ for all $X < H$ and $(M/J)(H) \neq 0$. This is equivalent to the conditions $f(M/J) = 0$ and $e(M/J) \neq 0$. Note that $f(M/J) = 0$ if and only if $AfM \subseteq J$, and that $e(M/J) \neq 0$ if and only if J does not contain AeM. Thus the sets $\mathcal{A}$ and $\mathcal{B}$ are equal.

Let J be an A-submodule of M and I be an eAe-submodule of eM. If J contains AfM then eJ contains $eAfM$, and conversely if I contains $eAfM$ then, by its definition, $(M :_e I)$ contains AfM. Therefore, it follows by part (1) of 4.3 that the maps $J \to eJ$ and $(M :_e I) \leftarrow I$ define a bijective correspondence between the maximal elements of the sets $\mathcal{B}$ and $\mathcal{C}$.

We finish the proof by showing the equality of the sets $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$: By the basis Theorem 2.1 it is clear that any element of eAf can be written as a linear combination of the elements of the form $t_{{}^gA}^H c_A^g r_A^X$ where $X < H$ so that ${}^gA < H$. Moreover, it is obvious that t_S^H is in eAf for any $S < H$. Consequently, $eAfM = b_H(M)$. The elements of the form $t_{{}^xB}^H c_B^x r_B^H$ with $B \neq H$ form a $\mathbb{K}$-basis of the two sided ideal I_H of eAe, see 2.4. This shows that $I_H eM = I_H M$ is in $b_H(M)$. Therefore,

$$I_H M \subseteq eAfM = b_H(M) \subseteq eM.$$

By the correspondence theorem, there is a bijection between the eAe-submodules of eM containing $eAfM$ and eAe-submodules of $M/I_H M$ containing $eAfM/I_H M$. As the ideal I_H annihilates the

eAe-module M/I_HM and as $eAe = B \oplus I_H$, the eAe-submodules of M/I_HM and the B-submodules of M/I_HM are the same. By another usage of the correspondence theorem, we see that the eAe-submodules of eM containing $eAfM$ and the B-submodules of eM containing $eAfM$ are the same. As $eAfM = b_H(M)$, the sets $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ are equal. $\square$

Let M be a $\mu_{\mathbb{K}}(G)$-module and H be a subgroup of G. A consequence of 4.4 is that there is an injection from the set of maximal $\mu_{\mathbb{K}}(G)$-submodules J of M such that $M/J \cong S^G_{H,V}$ for some simple $\mathbb{K}\overline{N}_G(H)$-module V to the set of maximal $\mathbb{K}\overline{N}_G(H)$-submodules of $\overline{M}(H)$. Indeed, by part (2) of 4.2 (or part (3) of 4.3) we see that the maps in 4.4 define a bijection between the maximal $\mu_{\mathbb{K}}(G)$-submodules J of M that satisfies the given condition in 4.4 and the maximal $\mathbb{K}\overline{N}_G(H)$-submodules $I/b_H(M)$ of $\overline{M}(H)$ that satisfies $AeM + (M :_e I) = M$, where $A = \mu_{\mathbb{K}}(G)$ and $e = t^H_H$.

Lemma 4.5. *Let M be a $\mu_{\mathbb{K}}(G)$-module, let H be a subgroup of G, and let $e = t^H_H$. Then:*

(1) *$I_HM \subseteq b_H(M)$ so that $b_H(M)$ is a $e\mu_{\mathbb{K}}(G)e$-submodule of $M(H)$ where I_H is the ideal of $e\mu_{\mathbb{K}}(G)e$ given in 2.4. In particular, any $\mathbb{K}\overline{N}_G(H)$-submodule of $M(H)$ containing $b_H(M)$ is an $e\mu_{\mathbb{K}}(G)e$-submodule of $M(H)$.*

(2) *If $H \nleqslant_G X$ then $(M :_e b_H(M))(X) = M(X)$, and if $H \leqslant_G X$ then*

$$\big(M :_e b_H(M)\big)(X) = \big\{x \in M(X)\colon\ c^g_{H^g} r^X_{H^g}(x) \in b_H(M),\ \forall g \in G \text{ with } H^g \leqslant X\big\}.$$

(3) *Let $\mathcal{X} = \{X \leqslant G\colon\ H \not< _G X\}$ and $I/b_H(M)$ be a $\mathbb{K}\overline{N}_G(H)$-submodule of $\overline{M}(H)$. Then, for any subset $\mathcal{Y}$ of $\mathcal{X}$ containing a G-conjugate of H we have*

$$AeM + (M :_e I) = Ae_{\mathcal{Y}}M + (M :_e I),$$

where $A = \mu_{\mathbb{K}}(G)$ and $e_{\mathcal{Y}}$ is the idempotent of A defined as in 3.18. *In particular, the evaluations of the functors $AeM + (M :_e I)$ and M at subgroups of G in $\mathcal{X}$ are all equal.*

Proof. (1) It is obtained in the proof of 4.4.

(2) As

$$\big(M :_e b_H(M)\big)(X) = \big\{x \in M(X)\colon\ t^H_H \mu_{\mathbb{K}}(G) t^X_X x \subseteq b_H(M)\big\},$$

by the basis Theorem 2.1 we see that $(M :_e b_H(M))(X)$ is the set of all elements $x \in M(X)$ satisfying $t^H_{{}^gJ} c^g_J r^X_J(x) \in b_H(M)$ for all $g \in G$ and all $J \leqslant H^g \cap X$. Note that if ${}^gJ < H$ then this condition is satisfied trivially for all $x \in M(X)$. Thus, the result follows.

(3) Since $(M :_e b_H(M)) \subseteq (M :_e I)$, it follows by part (2) that $(M :_e I)(Y) = M(Y)$ for all $Y \in \mathcal{Y}$ with $Y \neq_G H$. If $Y =_G H$ then it is clear that $(AeM)(Y) = M(Y)$ (because $e = t^H_H$). Therefore,

$$AeM \subseteq Ae_{\mathcal{Y}}M \subseteq AeM + (M :_e I),$$

from which the result follows. $\square$

Corollary 4.6. *Let M be a $\mu_{\mathbb{K}}(G)$-module and H be a subgroup of G. Put $e = t^H_H$. Then:*

(1) *The maps $J \to J(H)$ and $(M :_e I) \leftarrow I$ define a bijective correspondence between the maximal $\mu_{\mathbb{K}}(G)$-submodules J of M such that $M/J \cong S^G_{H,V}$ for some simple $\mathbb{K}\overline{N}_G(H)$-module V and the maximal $\mathbb{K}\overline{N}_G(H)$-submodules $I/b_H(M)$ of $\overline{M}(H)$ that satisfies*

$$M(X)=\sum_{g\in G} t^X_{X\cap {}^gH}\big(M\big(X\cap {}^gH\big)\big)+\big\{x\in M(X)\colon c^g_{H^g} r^X_{H^g}(x)\in I,\ \forall g\in G,\ H^g\leqslant X\big\}$$

for all $X\leqslant G$ with $H<X$.

(2) *Let M be a semisimple $\mu_{\mathbb{K}}(G)$-module. Then, the maps $J\to J(H)$ and $(M:_e I)\leftarrow I$ define a bijective correspondence between the maximal $\mu_{\mathbb{K}}(G)$-submodules J of M such that $M/J\cong S^G_{H,V}$ for some simple $\mathbb{K}\overline{N}_G(H)$-module V and the maximal $\mathbb{K}\overline{N}_G(H)$-submodules $I/b_H(M)$ of $\overline{M}(H)$.*

Proof. Let $A=\mu_{\mathbb{K}}(G)$, $\mathcal{Y}=\{Y\leqslant G\colon\ Y\leqslant_G H\}$, and let the idempotent $e'=e_{\mathcal{Y}}$ be defined as in 3.18.

(1) Let $I/b_H(M)$ be a $\mathbb{K}\overline{N}_G(H)$-submodule of $\overline{M}(H)$. It follows by part (3) of 4.5 that the A-modules

$$AeM+(M:_e I)\quad\text{and}\quad Ae'M+(M:_e I)$$

are equal. Since $\mathcal{Y}$ is closed under taking subgroups and taking G-conjugates, we see easily by using the basis Theorem 2.1 that

$$(Ae'M)(X)=\sum_{Y\in\mathcal{Y}\colon\ Y\leqslant X} t^X_Y\big(M(Y)\big)$$

for any $X\leqslant G$. Part (3) of 4.5 implies that the evaluations of

$$AeM+(M:_e I)\quad\text{and}\quad M$$

at subgroups X of G for which $H\nless_G X$ are all equal. Thus, to justify that

$$AeM+(M:_e I)=M$$

it is enough to see that

$$(Ae'M)(X)+(M:_e I)(X)=M(X)$$

for all X with $H<_G X\leqslant G$. As the conjugation maps c^g_X of M are $\mathbb{K}$-space isomorphism, it is enough to see the equality of the above evaluations at subgroups X satisfying $H<X\leqslant G$.

Let $H<X\leqslant G$. If $Y\in\mathcal{Y}$ with $Y\leqslant X$ then there is a $g\in G$ such that $Y\leqslant X\cap {}^gH\in\mathcal{Y}$. By the transitivity property (M_1) of the trace maps on M (see the definition of a Mackey functor given in Section 2) we have

$$t^X_Y\big(M(Y)\big)\subseteq t^X_{X\cap {}^gH}\big(M\big(X\cap {}^gH\big)\big).$$

Therefore,

$$(Ae'M)(X)=\sum_{g\in G} t^X_{X\cap {}^gH}\big(M\big(X\cap {}^gH\big)\big).$$

Moreover, since $b_H(M)\subseteq I$, we see as in the proof of part (2) of 4.5 that

$$(M:_e I)(X)=\big\{x\in M(X)\colon\ c^g_{H^g} r^X_{H^g}(x)\in I,\ \forall g\in G,\ H^g\leqslant X\big\}.$$

The result now follows by the explanation given at the beginning of 4.5.

(2) By the explanation given at the beginning of 4.5, it suffices to prove that

$$AeM + (M :_e I) = M$$

for any maximal $\mathbb{K}\overline{N}_G(H)$-submodule $I/b_H(M)$ of $\overline{M}(H)$. Indeed, this is true for any (not necessarily maximal) $\mathbb{K}\overline{N}_G(H)$-submodule $I/b_H(M)$. To see this, we first note by part (5) of 4.1 that $(M/(M :_e I) :_e 0) = 0$. As M is semisimple, part (8) of 4.1 implies the result. □

The condition on I given in part (1) of 4.6 becomes slightly simpler if we assume that H is normal in G. Using 3.13 we see that the existence of a maximal subfunctor J of M such that $M/J \cong S^G_{H,V}$ for some simple $\mathbb{K}\overline{N}_G(H)$-module V is equivalent to the existence of a maximal subfunctor J' of $\downarrow^G_{N_G(H)} M$ such that $(\downarrow^G_{N_G(H)} M)/J' \cong S^{N_G(H)}_{H,V}$.

Corollary 4.7. *Let M be a $\mu_{\mathbb{K}}(G)$-module and H be a subgroup of G. Put $M' = \downarrow^G_{N_G(H)} M$ and $e = t^H_H$. Then:*

(1) *The maps $J \to (M' :_e J(H))$ and $(M :_e J'(H)) \leftarrow J'$ define a bijective correspondence between the largest elements of the set of all subfunctors J of M whose quotient functor M/J has H as a minimal subgroup and the largest elements of the set of all subfunctors J' of M' whose quotient functor M'/J' has H as a minimal subgroup.*
(2) *The map $J \to (M' :_e J(H))$ define an injection from the set of all maximal $\mu_{\mathbb{K}}(G)$-submodules J of M such that H is a minimal subgroup of the simple functor M/J to the set of all maximal $\mu_{\mathbb{K}}(N_G(H))$-submodules J' of M' such that H is a minimal subgroup of the simple functor M'/J'.*
(3) *For any maximal $\mu_{\mathbb{K}}(G)$-submodule J of M such that H is a minimal subgroup of the simple functor M/J, there is a maximal $\mu_{\mathbb{K}}(N_G(H))$-submodule J' of M' such that H is a minimal subgroup of the simple functor M'/J' and $J = (M :_e J'(H))$.*

Proof. (1) This follows from 4.4 because, for any $H \leqslant K \leqslant G$, it follows by the definition of the Brauer quotient that

$$b_H(M) = b_H\big(\downarrow^G_K M\big) \quad \text{and} \quad \overline{M}(H) = \overline{\big(\downarrow^G_K M\big)}(H).$$

(2) and (3) Let $A = \mu_{\mathbb{K}}(G)$, $B = \mu_{\mathbb{K}}(N_G(H))$, and $L = N_G(H)$. Let J be a maximal A-submodule of M such that H is a minimal subgroup of the simple A-module M/J. Then M/J must be isomorphic to a simple functor of the form $S^G_{H,V}$. Using 3.13 we see that the multiplicity of the simple B-module $S^L_{H,V}$ in the head of $\downarrow^G_L (M/J) \cong M'/(\downarrow^G_L J)$ is nonzero (indeed one) where the isomorphism of the B-modules follows from the exactness of the functor $\downarrow^G_L$. Therefore there is a maximal B-submodule J' of M' containing $\downarrow^G_L J$ such that M'/J' is isomorphic to $S^L_{H,V}$. In particular, $J(H) = J'(H)$. Moreover, part (4) of 4.2 implies that $J' = (M' :_e J'(H))$ and $J = (M :_e J(H))$. Using the equality $J(H) = J'(H)$ we obtain that $J' = (M' :_e J(H))$ and $J = (M :_e J'(H))$.

As $(M' :_e J(H))$ is equal to the maximal B-submodule J' of M', part (2) follows. As the maximal B-submodule J' of M' satisfies $J = (M :_e J'(H))$, part (3) follows. □

Lemma 4.8. *Let M be a $\mu_{\mathbb{K}}(G)$-module and H be a normal subgroup of G. Put $A = \mu_{\mathbb{K}}(G)$ and $e = t^H_H$. The following conditions are equivalent for any maximal $\mathbb{K}(G/H)$-submodule $\bar{I} = I/b_H(M)$ of $\overline{M}(H)$:*

(i) *$(M :_e I)$ is a maximal $\mu_{\mathbb{K}}(G)$-submodule of M.*
(ii) *$AeM + (M :_e I) = M$.*
(iii) *$M(X) = t^X_H(M(H)) + \{x \in M(X)\colon\ r^X_H(x) \in I\}$ for all $X \leqslant G$ with $H < X$.*

(iv) *For all* $X \leqslant G$ *with* $H < X$,

$$r_H^X\big(M(X)\big) \subseteq \left(\sum_{gH \subseteq X} c_H^g\right) M(H) + I.$$

(v) *For all* $X \leqslant G$ *with* $H < X$,

$$\big(r_H^X\big(M(X)\big) + b_H(M)\big)/b_H(M) \subseteq \left(\sum_{gH \subseteq X} c_H^g\right) \overline{M}(H) + \bar{I}.$$

(vi) *There is a simple* $\mathbb{K}(G/H)$*-module* U *and a nonzero* $\alpha \in \mathrm{Hom}_{\mathbb{K}(G/H)}(\overline{M}(H), U)$ *with kernel equal to* $\bar{I}$ *and such that*

$$\alpha \circ \pi_H \circ r_H^X\big(M(X)\big) \subseteq \left(\sum_{gH \subseteq X} c_H^g\right) U$$

for all $X \leqslant G$ *with* $H < X$*, where* $\pi_H : M(H) \to M(H)/b_H(M)$ *is the natural epimorphism.*

Proof. (i), (ii), and (iii) are equivalent: Follows from 4.2 and 4.6.

(iv) is equivalent to (v): Clear.

(iii) implies (iv): Take any $x \in M(X)$. Then $x = t_H^X(a) + b$ for some $a \in M(H)$ and $b \in M(X)$ with $r_H^X(b) \in I$. By the Mackey axiom

$$r_H^X(x) = r_H^X t_H^X(a) + r_H^X(b) = \left(\sum_{gH \subseteq X} c_H^g\right) a + r_H^X(b) \in \left(\sum_{gH \subseteq X} c_H^g\right) M(H) + I.$$

(iv) implies (iii): Take any $x \in M(X)$. Then, there is a $u \in M(H)$ and $v \in I$ such that

$$r_H^X(x) = \left(\sum_{gH \subseteq X} c_H^g\right) u + v.$$

By the Mackey axiom

$$r_H^X(x) = r_H^X t_H^X(u) + v, \quad \text{implying that} \quad r_X^H\big(x - t_H^X(u)\big) = v \in I.$$

Consequently,

$$x = t_H^X(u) + \big(x - t_H^X(u)\big) \in t_H^X\big(M(H)\big) + \big\{x \in M(X){:}\ r_H^X(x) \in I\big\}.$$

(v) implies (vi): Put $U = \overline{M}(H)/\bar{I}$ and let $\alpha : \overline{M}(H) \to U$ be the natural surjection. Then, U is a simple $\mathbb{K}(G/H)$-module and α is a (nonzero) $\mathbb{K}(G/H)$-module epimorphism with kernel equal to $\bar{I}$. Moreover, using (v), we have

$$\alpha \circ \pi_H \circ r_H^X\big(M(X)\big) = \alpha\big(\big(r_H^X\big(M(X)\big) + b_H(M)\big)/b_H(M)\big) \subseteq \left(\sum_{gH \subseteq X} c_H^g\right) U + \alpha(\bar{I}) = \left(\sum_{gH \subseteq X} c_H^g\right) U.$$

(vi) implies (iv): Take $x \in M(X)$. As a result of (vi) there is a $y \in M(H)$ such that

$$\alpha\big(r_H^X(x) + b_H(M)\big) = \alpha \circ \pi_H \circ r_H^X(x) = \Big(\sum_{gH\subseteq X} c_H^g\Big)\alpha\big(y + b_H(M)\big).$$

This shows that

$$r_H^X(x) - \Big(\sum_{gH\subseteq X} c_H^g\Big)y + b_H(M) \in \operatorname{Ker}\alpha = \bar{I} \quad \text{implying} \quad r_H^X(x) \in \Big(\sum_{gH\subseteq X} c_H^g\Big)M(H) + I. \qquad \square$$

Corollary 4.9. *(See [11, (15.4) Proposition].) Let M be a $\mu_{\mathbb{K}}(G)$-module, let H be a subgroup of G, and let U be a simple $\mathbb{K}\overline{N}_G(H)$-module. Then,* $\operatorname{Hom}_{\mu_{\mathbb{K}}(G)}(M, S_{H,U}^G) \neq 0$ *if and only if there is a nonzero* $\alpha \in \operatorname{Hom}_{\mathbb{K}\overline{N}_G(H)}(\overline{M}(H), U)$ *such that*

$$\alpha \circ \pi_H \circ r_H^X\big(M(X)\big) \subseteq \Big(\sum_{gH\subseteq X} c_H^g\Big)U$$

for all $X \leqslant G$ with $H < X \leqslant N_G(H)$, where $\pi_H : M(H) \to M(H)/b_H(M)$ is the natural epimorphism.

Proof. By 3.13 we may assume that H is normal in G, because $\overline{M}(H) = \overline{(\downarrow_K^G M)}(H)$ for any $H \leqslant K \leqslant G$. Put $e = t_H^H$.

Suppose that $\operatorname{Hom}_{\mu_{\mathbb{K}}(G)}(M, S_{H,U}^G) \neq 0$. There is a maximal subfunctor J of M such that $M/J \cong S_{H,U}^G$. Moreover, $U \cong M(H)/I \cong \overline{M}(H)/\bar{I}$ as $\mathbb{K}(G/H)$-modules, where $I = J(H)$. It follows by 4.6 that $J = (M :_e I)$ and that $\bar{I}$ is a maximal $\mathbb{K}(G/H)$-submodule of $\overline{M}(H)$ satisfying the equivalent conditions (in particular (vi)) of 4.8. Thus there is a simple $\mathbb{K}(G/H)$-module U' and a (nonzero) $\mathbb{K}(G/H)$-module epimorphism $\alpha' : \overline{M}(H) \to U'$ with kernel equal to $\bar{I}$ so that $U \cong U'$, and such that

$$\alpha' \circ \pi_H \circ r_H^X\big(M(X)\big) \subseteq \Big(\sum_{gH\subseteq X} c_H^g\Big)U'.$$

Let $f : U' \to U$ be a $\mathbb{K}(G/H)$-module isomorphism. Put $\alpha = f \circ \alpha'$ which is a nonzero element of $\operatorname{Hom}_{\mathbb{K}(G/H)}(\overline{M}(H), U)$. Now,

$$\alpha \circ \pi_H \circ r_H^X\big(M(X)\big) = f \circ \alpha' \circ \pi_H \circ r_H^X\big(M(X)\big) \subseteq \Big(\sum_{gH\subseteq X} c_H^g\Big)f(U') = \Big(\sum_{gH\subseteq X} c_H^g\Big)U.$$

Conversely, assume that there is a nonzero $\alpha \in \operatorname{Hom}_{\mathbb{K}(G/H)}(\overline{M}(H), U)$ satisfying the required conditions. Letting $\bar{I} = \operatorname{Ker}\alpha$, we see that $\bar{I}$ is a maximal $\mathbb{K}(G/H)$-submodule of $\overline{M}(H)$ satisfying the condition (vi) of 4.8 and such that $\overline{M}(H)/I \cong U$. Thus $J = (M :_e I)$ is a maximal $\mu_{\mathbb{K}}(G)$-submodule of M, and H is a minimal subgroup of M/J, and $J(H) = I$ so that $M/J \cong S_{H,U}^G$. $\square$

5. Restriction kernels and minimal subfunctors

Let M be a $\mu_{\mathbb{K}}(G)$-module and H be a subgroup of G. This section deals with the simple $\mu_{\mathbb{K}}(G)$-submodules of M and the $\mathbb{K}\overline{N}_G(H)$-submodules of the coordinate module $M(H)$ of M. We want to obtain results similar to the ones obtained in the previous section, and give some refinements. We usually skip the proofs of the similar results.

We begin with the following whose proof is routine.

Lemma 5.1. *Let A be a finite dimensional $\mathbb{K}$-algebra and e be a nonzero idempotent of A. Suppose that V is a nonzero A-module, S is an A-submodule of V, and T is an eAe-submodule of eV. Then:*

(1) *T is a simple eAe-submodule of eV if and only if AT is a smallest element of the set of all A-submodules of V not contained in $(V :_e 0)$.*
(2) *AT is a simple A-submodule of V if and only if T is a simple eAe-submodule of eV and $(AT :_e 0) = 0$.*
(3) *S is a smallest element of the set of all A-submodules of V not contained in $(V :_e 0)$ if and only if eS is a simple eAe-submodule of eV and $S = AeS$.*
(4) *$S = AeS$ if and only if S has no nonzero quotient module annihilated by e.*

From 5.1 the following is immediate.

Proposition 5.2. *Let A be a finite dimensional $\mathbb{K}$-algebra and e be a nonzero idempotent of A. Suppose that V is a nonzero A-module. Then:*

(1) *The maps $S \to eS$ and $AT \leftarrow T$ define a bijective correspondence between the smallest elements of the set of all A-submodules of V not contained in $(V :_e 0)$ and the simple eAe-submodules of eV.*
(2) *The maps $S \to eS$ and $AT \leftarrow T$ define a bijective correspondence between the simple A-submodules of V that are contained in $(V :_{1-e} 0)$ (so, necessarily not contained in $(V :_e 0)$) and the simple eAe-submodules of eV that are contained in $e(V :_{1-e} 0)$.*
(3) *The maps $S \to eS$ and $AT \leftarrow T$ define a bijective correspondence between the simple A-submodules of V that are not contained in $(V :_e 0)$ and the simple eAe-submodules of eV that satisfy $(AT :_e 0) = 0$.*

We now need to recall the notion of the restriction kernel of a Mackey functor, see [9,12]. Let M be a $\mu_{\mathbb{K}}(G)$-module and H be a subgroup of G. By the restriction kernel of M at H we mean the following module

$$\underline{M}(H) = \bigcap_{J<H} \operatorname{Ker}\bigl(r_J^H : M(H) \to M(J)\bigr).$$

It is clear that $\underline{M(H)}$ is a $\mathbb{K}\overline{N}_G(H)$-submodule of $M(H)$. Moreover, there is a $\mathbb{K}\overline{N}_G(H)$-module isomorphism $\underline{M}(H) \cong (\overline{(M^*)}(H))^*$ obtained by taking $\mathbb{K}$-duals, see [12]. Thus, every result concerning Brauer quotients has a dual result concerning restriction kernels. In this section we obtain these dual results and refine them. However we will not make use of this duality property here.

Lemma 5.3. *Let M be a $\mu_{\mathbb{K}}(G)$-module, let H be a subgroup of G, and let T be a $\mathbb{K}\overline{N}_G(H)$-submodule of $\underline{M}(H)$. Put $A = \mu_{\mathbb{K}}(G)$ and $e = t_H^H$. Then:*

(1) *The ideal I_H of eAe defined in 2.4 annihilates $\underline{M}(H)$ so that $\underline{M}(H)$ is also an eAe-submodule of $M(H)$ whose $\mathbb{K}\overline{N}_G(H)$-submodules and eAe-submodules are the same.*
(2) *Let $\mathcal{X}$ be a set of subgroups of G. If $\{X \leqslant G\colon X < H\} \subseteq \mathcal{X} \subseteq \{X \leqslant G\colon H \nleqslant_G X\}$, then $(M :_{e_{\mathcal{X}}} 0)(H) = \underline{M}(H)$, where $e_{\mathcal{X}}$ is the idempotent of A defined as in 3.18.*
(3) *If $H \nleqslant_G X$ then $(AT)(X) = 0$, and if $H \leqslant_G X$ then*

$$(AT)(X) = \sum_{g\in G\colon {}^gH\leqslant X} t_{{}^gH}^X c_H^g(T).$$

(4) *If $H \nless_G X$ then $(AT :_e 0)(X) = 0$, and if $H <_G X$ then*

$$(AT :_e 0)(X) = \left(\sum_{g\in G\colon {}^gH\leqslant X} t_{{}^gH}^X c_H^g(T)\right) \cap \left(\bigcap_{g\in G\colon {}^gH\leqslant X} \operatorname{Ker}\bigl(r_{{}^gH}^X : M(X) \to M({}^gH)\bigr)\right).$$

Proof. (1) Follows from 2.4 because any element of I_H is a linear combination of elements of the form $t^H_{{}^gJ} c^g_J r^H_J$ with $J \neq H$.

(2) By its definition

$$(M :_{e_{\mathcal{X}}} 0)(H) = \{x \in M(H):\ t^X_X \mu_{\mathbb{K}}(G) t^H_H x = 0,\ \forall X \in \mathcal{X}\}.$$

The basis Theorem 2.1 implies the result. Because, $J < H$ for any basis element $t^X_{{}^gJ} c^g_J r^H_J$ of $t^X_X \mu_{\mathbb{K}}(G) t^H_H$, and because if $J < H$ then $J \in \mathcal{X}$ so that r^H_J is in $t^X_X \mu_{\mathbb{K}}(G) t^H_H$ for some $X \in \mathcal{X}$.

(3) It is clear that

$$(AT)(X) = t^X_X \mu_{\mathbb{K}}(G) t^H_H T.$$

As $T \subseteq \underline{M}(H)$, if $J < H$ then r^H_J annihilates T. The result follows by the basis Theorem 2.1.

(4) Part (3) implies that if $H \not\leqslant_G X$ then $(AT :_e 0)(X) \subseteq (AT)(X) = 0$. Moreover,

$$(AT :_e 0)(H) = e(AT :_e 0) = 0.$$

So we now assume that $H <_G X$. For any $g \in G$ and any $J \leqslant H^g \cap X$, we see that if $x \in (AT)(X)$ then

$$t^H_{{}^gJ} c^g_J r^X_J x \in t^H_{{}^gJ}\big((AT)({}^gJ)\big) = 0$$

in the case ${}^gJ \neq H$. Thus, as the conjugation maps $c^g_{H^g}$ of M are bijections, from the basis Theorem 2.1 we obtain

$$\begin{aligned}
(AT :_e 0)(X) &= \{x \in (AT)(X):\ t^H_H \mu_{\mathbb{K}}(G) t^X_X x = 0\} \\
&= \{x \in (AT)(X):\ c^g_{H^g} r^X_{H^g}(x) = 0,\ \forall g \in G,\ H^g \leqslant X\} \\
&= \{x \in (AT)(X):\ r^X_{H^g}(x) = 0,\ \forall g \in G,\ H^g \leqslant X\} \\
&= (AT)(X) \cap \{x \in M(X):\ r^X_{H^g}(x) = 0,\ \forall g \in G,\ H^g \leqslant X\}.
\end{aligned}$$

The result now follows from part (3). □

Theorem 5.4. *Let M be a $\mu_{\mathbb{K}}(G)$-module and H be a subgroup of G. Put $e = t^H_H$ and $A = \mu_{\mathbb{K}}(G)$. Then, the maps $S \to S(H)$ and $AT \leftarrow T$ define a bijective correspondence between the smallest elements of the set of all subfunctors S of M having H as a minimal subgroup and the simple $\mathbb{K}\overline{N}_G(H)$-submodules T of $\underline{M}(H)$. In particular, $\underline{M}(H) = 0$ if and only if M has no subfunctor having H as a minimal subgroup.*

Proof. We argue as in the proof of 4.4. Let $B = \mathbb{K}\overline{N}_G(H)$, $\mathcal{X} = \{X \leqslant G:\ X < H\}$, and $f = e_{\mathcal{X}}$ be the idempotent of A defined as in 3.18.

We also define four sets $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ as follows. $\mathcal{A}$ is the set of all subfunctors of M having H as a minimal subgroup, $\mathcal{B}$ is the set of all A-submodules of M contained in $(M :_f 0)$ but not contained in $(M :_e 0)$, $\mathcal{C}$ is the set of all eAe-submodules of eM contained in $e(M :_f 0)$, and $\mathcal{D}$ is the set of all B-submodules of $\underline{M}(H)$.

It is easy to see that the sets $\mathcal{A}$ and $\mathcal{B}$ are equal. Moreover, it follows by 5.3 that the sets $\mathcal{C}$ and $\mathcal{D}$ are equal. Because, 5.3 implies that $e(M :_f 0) = \underline{M}(H)$ and that I_H annihilates $\underline{M}(H)$.

Now the result follows from part (1) of 5.2 which shows that the maps $S \to eS$ and $AT \leftarrow T$ define a bijective correspondence between the minimal elements of the sets $\mathcal{B}$ and $\mathcal{C}$. □

Given a $\mu_{\mathbb{K}}(G)$-module M and a subgroup H of G, the previous result implies that there is an injection from the set of simple $\mu_{\mathbb{K}}(G)$-submodules S of M isomorphic to $S^G_{H,V}$ for some simple

$\mathbb{K}\overline{N}_G(H)$-module V to the set of simple $\mathbb{K}\overline{N}_G(H)$-submodules of $\underline{M}(H)$. Indeed, we see by part (3) of 5.2 that the maps in 5.4 define a bijection between the simple $\mu_{\mathbb{K}}(G)$-submodules S of M having H as a minimal subgroup and the simple $\mathbb{K}\overline{N}_G(H)$-submodules T of $\underline{M}(H)$ that satisfies $(AT :_e 0) = 0$, where $A = \mu_{\mathbb{K}}(G)$ and $e = t_H^H$.

Remark 5.5. Let A be a finite dimensional $\mathbb{K}$-algebra and e be a nonzero idempotent of A. If V is a semisimple A-module, then $(AT :_e 0) = 0$ for any eAe-submodule T of eV.

Proof. As V is semisimple, $(AT :_e 0) \oplus W = AT$ for some A-submodule W of AT. Multiplying both sides with e we get $eW = T$, implying that $AT = AeW \subseteq W$. Hence, $(AT :_e 0) = 0$. □

Corollary 5.6. *Let M be a $\mu_{\mathbb{K}}(G)$-module and H be a subgroup of G. Put $A = \mu_{\mathbb{K}}(G)$ and $e = t_H^H$. Then:*

(1) *The maps $S \to S(H)$ and $AT \leftarrow T$ define a bijective correspondence between the simple $\mu_{\mathbb{K}}(G)$-submodules S of M isomorphic to $S_{H,V}^G$ for some simple $\mathbb{K}\overline{N}_G(H)$-module V and the simple $\mathbb{K}\overline{N}_G(H)$-submodules T of $\underline{M}(H)$ that satisfies*

$$0 = \Bigg(\sum_{g \in G:\ {}^gH \leqslant X} t_{{}^gH}^X c_H^g(T)\Bigg) \cap \Bigg(\bigcap_{g \in G:\ {}^gH \leqslant X} \operatorname{Ker}\big(r_{{}^gH}^X : M(X) \to M\big({}^gH\big)\big)\Bigg)$$

for all $X \leqslant G$ with $H < X$.

(2) *Let M be a semisimple $\mu_{\mathbb{K}}(G)$-module. Then, the maps $S \to S(H)$ and $AT \leftarrow T$ define a bijective correspondence between the simple $\mu_{\mathbb{K}}(G)$-submodules S of M isomorphic to $S_{H,V}^G$ for some simple $\mathbb{K}\overline{N}_G(H)$-module V and the simple $\mathbb{K}\overline{N}_G(H)$-submodules T of $\underline{M}(H)$.*

Proof. Follows from 5.3, 5.5, and from the explanation given at the beginning of 5.5. □

Corollary 5.7. *Let M be a $\mu_{\mathbb{K}}(G)$-module and H be a subgroup of G. Put $M' = \downarrow_{N_G(H)}^G M$, $e = t_H^H$, $A = \mu_{\mathbb{K}}(G)$, and $B = \mu_{\mathbb{K}}(N_G(H))$. Then:*

(1) *The maps $S \to BeS$ and $AeS' \leftarrow S'$ define a bijective correspondence between the smallest elements of the set of all subfunctors S of M having H as a minimal subgroup and the smallest elements of the set of all subfunctors S' of M' having H as a minimal subgroup.*
(2) *The map $S \to BeS$ define an injection from the set of all simple $\mu_{\mathbb{K}}(G)$-submodules S of M such that H is a minimal subgroup of S to the set of all simple $\mu_{\mathbb{K}}(N_G(H))$-submodules S' of M' such that H is a minimal subgroup of S'.*
(3) *For any simple $\mu_{\mathbb{K}}(G)$-submodule S of M such that H is a minimal subgroup of S, there is a simple $\mu_{\mathbb{K}}(N_G(H))$-submodules S' of M' such that H is a minimal subgroup of S' and $S = AeS'$.*

Proof. (1) This can be deduced by arguing as in the proof of part (1) of 4.7.

(2) and (3) Let $K = N_G(H)$. Let S be a simple A-submodule of M having H as a minimal subgroup. S must be isomorphic to a simple functor of the form $S_{H,V}^G$. Using 3.13 we see that there is a simple B-submodule S' of $\downarrow_K^G S \subseteq \downarrow_K^G M$ isomorphic to $S_{H,V}^K$. In particular, $eS = eS' \neq 0$. Moreover, $S = AeS$ and $S' = BeS'$ by the simplicity of the A-module S and the B-module S'. Using the equality $eS = eS'$, we obtain that $S = AeS'$ (proving part (3)) and that $S' = BeS$ (proving part (2)). □

The condition on T given in part (1) of 5.6 becomes simpler if we assume that H is normal in G. The results 5.1 and 5.6, and the Mackey axiom imply the following.

Lemma 5.8. *Let M be a $\mu_{\mathbb{K}}(G)$-module and H be a normal subgroup of G. Put $A = \mu_{\mathbb{K}}(G)$ and $e = t_H^H$. The following conditions are equivalent for any simple $\mathbb{K}(G/H)$-submodule T of $\underline{M}(H)$:*

(i) *AT is a simple $\mu_{\mathbb{K}}(G)$-submodule of M.*
(ii) *$(AT :_e 0) = 0$.*
(iii) *$t_H^X(T) \cap \operatorname{Ker}(r_H^X : M(X) \to M(H)) = 0$ for all $X \leqslant G$ with $H < X$.*
(iv) *For all $X \leqslant G$ with $H < X$,*

$$x \in T, \quad \Big(\sum_{gH \subseteq X} c_H^g\Big) x = 0 \quad \textit{implies} \quad t_H^X(x) = 0.$$

(v) *There is a simple $\mathbb{K}(G/H)$-module U and a nonzero $\beta \in \operatorname{Hom}_{\mathbb{K}(G/H)}(U, \underline{M}(H))$ with image equal to T and such that*

$$\Big\{u \in U \colon \Big(\sum_{gH \subseteq X} c_H^g\Big) u = 0\Big\} \subseteq \operatorname{Ker}\big(t_H^X \circ \iota_H \circ \beta\big)$$

for all $X \leqslant G$ with $H < X$, where $\iota_H : \underline{M}(H) \to M(H)$ is the inclusion.

A justification similar to the proof of 4.9 can be given for the following result.

Corollary 5.9. *Let M be a $\mu_{\mathbb{K}}(G)$-module, let H be a subgroup of G, and let U be a simple $\mathbb{K}\overline{N}_G(H)$-module. Then, $\operatorname{Hom}_{\mu_{\mathbb{K}}(G)}(S_{H,U}^G, M) \neq 0$ if and only if there is a nonzero element β of $\operatorname{Hom}_{\mathbb{K}\overline{N}_G(H)}(U, \underline{M}(H))$ such that*

$$\Big\{u \in U \colon \Big(\sum_{gH \subseteq X} c_H^g\Big) u = 0\Big\} \subseteq \operatorname{Ker}\big(t_H^X \circ \iota_H \circ \beta\big)$$

for all $X \leqslant G$ with $H < X \leqslant N_G(H)$, where $\iota_H : \underline{M}(H) \to M(H)$ is the inclusion.

The result 5.8 contains some equivalent conditions to be checked for all $X \leqslant G$ with $H < X$. We next observe that we do not need to check them for all such X.

Lemma 5.10. *Let $\mathbb{K}$ be of characteristic $p > 0$. Suppose that M is a $\mu_{\mathbb{K}}(G)$-module and H is a normal subgroup of G. The following conditions are equivalent for any simple $\mathbb{K}(G/H)$-submodule T of $\underline{M}(H)$:*

(i) *The $\mu_{\mathbb{K}}(G)$-submodule of M generated by T is simple.*
(ii) *For any nontrivial p-subgroup X/H of G/H,*

$$x \in T, \quad \Big(\sum_{gH \subseteq X} c_H^g\Big) x = 0 \quad \textit{implies} \quad t_H^X(x) = 0.$$

Proof. Condition (i) is equivalent to the condition (iv) of 5.8. So, it suffices to see that part (ii) of the present result implies Part (iv) of 5.8. Let $Y \leqslant G$ with $H < Y$, and let $y \in T$ satisfying $(\sum_{gH \subseteq Y} c_H^g) y = 0$. We need to show that $t_H^Y(y) = 0$. Let X/H be a Sylow p-subgroup of Y/H. Using the axioms in the definition of a Mackey functor we see that

$$0 = \Big(\sum_{gH \subseteq Y} c_H^g\Big) y = r_H^Y t_H^Y(y) = r_H^X r_X^Y t_H^Y(y) = \sum_{Xg \subseteq Y} r_H^X t_H^X c_H^g(y) = \Big(\sum_{gH \subseteq X} c_H^g\Big) x,$$

where $x = \sum_{Xg \subseteq Y} c_H^g(y) \in T$. As X/H is a (nontrivial) p-subgroup of G/H, we must have that $0 = t_H^X(x)$, which implies $0 = t_H^Y(x) = |Y : X| t_H^Y(y)$. This gives that $t_H^Y(y) = 0$ because $|Y : X|$ is not divisible by p. $\square$

Remark 5.11. Let A be a finite dimensional $\mathbb{K}$-algebra and e be a nonzero idempotent of A. Let V be an A-module and let $W_1, W_2, \ldots, W_n$ be eAe-submodules of eV. Suppose that the A-submodules $AW_1, AW_2, \ldots, AW_n$ of V are all simple. If the sum of $W_1, W_2, \ldots, W_n$ is direct then the sum of $AW_1, AW_2, \ldots, AW_n$ is direct.

Proof. Suppose that the sum of $AW_1, AW_2, \ldots, AW_n$ is not direct. Therefore one of these simple A-modules must be in the sum of the others, say $AW_i \subseteq \sum_{j:\ j \neq i} AW_j$. Multiplying by e we obtain that $W_i \subseteq \sum_{j:\ j \neq i} W_j$, which is not true because the sum of $W_1, W_2, \ldots, W_n$ is direct. $\square$

Theorem 5.12. *Let $\mathbb{K}$ be of characteristic $p > 0$. Suppose that M is a $\mu_{\mathbb{K}}(G)$-module, H is a subgroup of G, and U is a simple $\mathbb{K}\overline{N}_G(H)$-module. Then, the multiplicity of $S^G_{H,U}$ in* $\mathrm{Soc}(M)$ *is equal to the multiplicity of U in the socle of the following $\mathbb{K}\overline{N}_G(H)$-submodule of $\underline{M}(H)$:*

$$\bigcap_{X/H} \left\{ x \in \underline{M}(H):\ \left(\sum_{gH \subseteq X} c^g_H \right) x = 0 \implies t^X_H(x) = 0 \right\}$$

where X/H ranges over all nontrivial p-subgroups of $N_G(H)/H$.

Proof. It is easy to see that the subset of $\underline{M}(H)$ defined as the above intersection is indeed a $\mathbb{K}\overline{N}_G(H)$-submodule of $\underline{M}(H)$.

It follows from 3.13 that we may (and will do) assume that H is normal in G, because $\underline{M}(H) = \underline{(\downarrow^G_K M)(H)}$ for any $H \leqslant K \leqslant G$. Let n be the multiplicity of $S^G_{H,U}$ in the socle of M, and let m be the multiplicity of U in the socle of the above given submodule of $\underline{M}(H)$ for which we use the notation $M_0(H)$ here.

Let $A = \mu_{\mathbb{K}}(G)$, $B = \mathbb{K}\overline{N}_G(H)$, and $e = t^H_H$. There are n simple A-submodules $S_1, S_2, \ldots, S_n$ of M whose sum is direct and all of them are isomorphic to $S^G_{H,U}$. Therefore the A-submodule $S_1 \oplus S_2 \oplus \cdots \oplus S_n$ of M is a direct summand of $\mathrm{Soc}(M)$. By 5.6, 5.8 and 5.10 we know that each eS_i is a simple B-submodule of $M_0(H)$. As the multiplication by the idempotent e respects the direct sums we see that the B-submodule $eS_1 \oplus eS_2 \oplus \cdots \oplus eS_n$ of $M_0(H)$ is a direct summand of $\mathrm{Soc}(M_0(H))$. As each $eS_i = S_i(H)$ is isomorphic to the simple B-module U, we conclude that $n \leqslant m$.

Conversely, there are m simple B-submodules $T_1, T_2, \ldots, T_m$ of $M_0(H) \subseteq \underline{M}(H)$ whose sum is direct and all of them are isomorphic to U. By part (1) of 5.3 we know that each T_i is also a simple eAe-submodule of $\underline{M}(H) \subseteq eM$. Moreover, it follows by 5.10 that each of the A-submodules AT_i of M is simple. Therefore we may apply 5.11 to deduce that the sum of the A-submodules AT_1, $AT_2, \ldots, AT_m$ of M is direct so that $AT_1 \oplus AT_2 \oplus \cdots \oplus AT_m$ is a direct summand of $\mathrm{Soc}(M)$. By 5.4 each simple A-module AT_i has H as a minimal subgroup, and as $AT_i(H) = T_i \cong U$ all of them must be isomorphic to $S^G_{H,U}$. Consequently, $m \leqslant n$. $\square$

Corollary 5.13. *Let $\mathbb{K}$ be of characteristic $p > 0$. Suppose that M is a $\mu_{\mathbb{K}}(G)$-module, H is a subgroup of G, and U is a simple $\mathbb{K}\overline{N}_G(H)$-module.*

(1) *There is a simple subfunctor of M having H as a minimal subgroup if and only if there is a simple $\mathbb{K}\overline{N}_G(H)$-submodule T of $\underline{M}(H)$ satisfying the following condition for any nontrivial p-subgroup X/H of $N_G(H)/H$:*

$$x \in T, \quad \left(\sum_{gH \subseteq X} c^g_H \right) x = 0 \quad \textit{implies} \quad t^X_H(x) = 0.$$

(2) *The multiplicity of $S^G_{H,U}$ in* $\mathrm{Soc}(M)$ *is less than or equal to the multiplicity of U in* $\mathrm{Soc}(\underline{M}(H))$.

(3) *The multiplicity of $S^G_{H,U}$ in* $\mathrm{Soc}(M)$ *is greater than or equal to the multiplicity of U in the socle of the following $\mathbb{K}\overline{N}_G(H)$-submodule of $\underline{M}(H)$:*

$$\bigcap_{H<X\leqslant N_G(H):\ |X:H|=p} \mathrm{Ker}\big(t^X_H : \underline{M}(H) \to M(X)\big).$$

(4) *Suppose that $\overline{N}_G(H)$ is a p'-group. Then, the multiplicity of $S^G_{H,U}$ in* $\mathrm{Soc}(M)$ *is equal to the multiplicity of U in $\underline{M}(H)$.*

Proof. (1) and (2) They are immediate from 5.12.

(3) By 5.12, it is enough to observe that the submodule of $\underline{M}(H)$ given in this part is in the submodule of $\underline{M}(H)$ given in 5.12: Let x be an element of the submodule of $\underline{M}(H)$ given in this part. It follows for any $X/H \leqslant N_G(H)/H$ with $|X : H| = p$ that $t^X_H(x) = 0$. Therefore, for any nontrivial p-subgroup Y/H of $N_G(H)/H$, it follows by the transitivity of trace maps on M that $t^Y_H(x) = 0$. Hence, x is in the submodule of $\underline{M}(H)$ given in 5.12.

(4) Follows from 5.12, because in this case the index set of the intersection defining the given submodule of $\underline{M}(H)$ is empty so that the intersection is equal to the semisimple $\mathbb{K}\overline{N}_G(H)$-module $\underline{M}(H)$. □

Part (2) of the previous result cannot be improved in general, because there may be two isomorphic simple $\mathbb{K}\overline{N}_G(H)$-submodules of $\underline{M}(H)$ such that the only one of them satisfies the condition given in part (1).

Letting $K = N_G(H)$, the adjointness of the pairs $(\uparrow^G_K, \downarrow^G_K)$ and $(\mathrm{Inf}^K_{K/H}, L^-{}_{K/H})$ and the isomorphism given in 2.7 and the result 3.12 imply that the multiplicity of a simple $\mu_{\mathbb{K}}(G)$-module $S^G_{H,U}$ in the socle of M is equal to the multiplicity of $S^{K/H}_{H/H,U}$ in the socle of the $\mu_{\mathbb{K}}(K/H)$-module $L^-{}_{K/H} \downarrow^G_K M$. Therefore, part (4) of 5.13 follows also from part (2) of 5.6 (because the Mackey algebra $\mu_{\mathbb{K}}(K/H)$ is semisimple in this case, see [10]).

The next result indicates a case in which the multiplicities mentioned in part (3) of 5.13 become equal.

Proposition 5.14. *Let $\mathbb{K}$ be of characteristic $p > 0$. Suppose that M is a $\mu_{\mathbb{K}}(G)$-module, H is a subgroup of G, and U is a simple $\mathbb{K}\overline{N}_G(H)$-module. If all the elements of order p of $\overline{N}_G(H)$ acts on U trivially, then the multiplicity of $S^G_{H,U}$ in* $\mathrm{Soc}(M)$ *is equal to the multiplicity of U in the socle of the following $\mathbb{K}\overline{N}_G(H)$-submodule of $\underline{M}(H)$:*

$$\bigcap_{H<X\leqslant N_G(H):\ |X:H|=p} \mathrm{Ker}\big(t^X_H : \underline{M}(H) \to M(X)\big).$$

Proof. Let T be a simple $\mathbb{K}\overline{N}_G(H)$-submodule of $M_0(H)$ isomorphic to U where $M_0(H)$ denotes the submodule of $\underline{M}(H)$ defined in 5.12. If we show that T is in the $\mathbb{K}\overline{N}_G(H)$-submodule of $\underline{M}(H)$ defined in this result (which is a submodule of $M_0(H)$), then the result will follow by 5.12.

Take any X with $H < X \leqslant N_G(H)$ and $|X : H| = p$. As the $\mathbb{K}\overline{N}_G(H)$-modules T and U are isomorphic and as any element of $N_G(H)$ of order p acts on U trivially, $(\sum_{gH\subseteq X} c^g_H)T = 0$. This implies that $t^X_H(T) = 0$ because $T \subseteq M_0(H)$. □

If $\overline{N}_G(H)$ is a nilpotent group (or more generally, a group with normal Sylow p-subgroup), then (Clifford's theorem implies that) the hypothesis of 5.14 is satisfied for any simple $\mathbb{K}\overline{N}_G(H)$-module U. For another example, the hypothesis of 5.14 is satisfied for any group G and for any simple $\mathbb{K}\overline{N}_G(H)$-module U with $\dim_{\mathbb{K}} U = 1$.

We next want to give a dual version of 5.12. Thus, given a $\mu_{\mathbb{K}}(G)$-module M and a subgroup of H of G, we want to find a quotient module of the $\mathbb{K}\overline{N}_G(H)$-module $\overline{M}(H)$ such that the multiplicity of a simple $\mathbb{K}\overline{N}_G(H)$-module V in the head of it is equal to the multiplicity of the simple $\mu_{\mathbb{K}}(G)$-module $S^G_{H,V}$ in the head of M. For this end we first need some trivial remarks.

Remark 5.15. Let V be a finite dimensional $\mathbb{K}$-vector space. For any $\mathbb{K}$-subspaces A, B, and W of V:

(1) Let $\mathcal{A} = \{f \in \mathrm{Hom}_{\mathbb{K}}(V, \mathbb{K})\colon\ f(W) = 0\}$. Then

$$W = \bigcap_{f \in \mathcal{A}} \mathrm{Ker}(f : V \to \mathbb{K})$$

where $\mathrm{Ker}(f : V \to \mathbb{K})$ denotes the kernel of f.

(2) Let $\mathcal{B} = \{f \in \mathrm{Hom}_{\mathbb{K}}(V, \mathbb{K})\colon\ f(B) = 0 \implies f(A) = 0\}$. Then, $A \subseteq B + W$ if and only if

$$\bigcap_{f \in \mathcal{B}} \mathrm{Ker}(f : V \to \mathbb{K}) \subseteq W.$$

Proof. This result is trivial. □

Using the following result, one may obtain a refinement similar to 5.10 for the equivalent conditions given in 4.8. Let M be a $\mu_{\mathbb{K}}(G)$-module. For a restriction map r^Y_X on M, it may not be true that $r^Y_X(b_Y(M)) \subseteq b_X(M)$. So, in general, r^Y_X does not induce a well-defined map from $\overline{M}(Y)$ to $\overline{M}(X)$. However, we will use the notations $r^X_H t^X_H(\overline{M}(H))$ and $r^X_H(\overline{M}(X))$ in some of our later results to indicate the subspaces $(r^X_H t^X_H(M(H)) + b_H(M))/b_H(M)$ and $(r^X_H(M(X)) + b_H(M))/b_H(M)$ of $\overline{M}(H)$, respectively.

Lemma 5.16. *Let $\mathbb{K}$ be of characteristic $p > 0$. Suppose that M is a $\mu_{\mathbb{K}}(G)$-module and H is a normal subgroup of G. Put $A = \mu_{\mathbb{K}}(G)$ and $e = t^H_H$. The following conditions are equivalent for any maximal $\mathbb{K}(G/H)$-submodule $\overline{I} = I/b_H(M)$ of $\overline{M}(H)$:*

(i) *$(M :_e I)$ is a maximal $\mu_{\mathbb{K}}(G)$-submodule of M.*
(ii) *$M(X) = t^X_H(M(H)) + \{x \in M(X)\colon\ r^X_H(x) \in I\}$ for any nontrivial p-subgroup X/H of G/H.*
(iii) *$\bigcap_f \mathrm{Ker}(f : \overline{M}(H) \to \mathbb{K}) \subseteq \overline{I}$ for any nontrivial p-subgroup X/H of G/H, where f ranges over all elements of the set*

$$\bigl\{f \in \mathrm{Hom}_{\mathbb{K}}\bigl(\overline{M}(H), \mathbb{K}\bigr)\colon\ f\bigl(r^X_H t^X_H\bigl(\overline{M}(H)\bigr)\bigr) = 0 \implies f\bigl(r^X_H\bigl(\overline{M}(X)\bigr)\bigr) = 0\bigr\}.$$

Proof. (i) is equivalent to (ii): By the virtue of 4.8, it suffices to show that part (ii) of the present result implies Part (ii) of 4.8. Let Y/H be any nontrivial subgroup of G/H. Take any $y \in M(Y)$. We need to show that

$$y \in t^Y_H\bigl(M(H)\bigr) + \bigl\{y \in M(Y)\colon\ r^Y_H(y) \in I\bigr\}.$$

Let X/H be a Sylow p-subgroup of Y/H and let $n = 1/|Y : X|$. As X/H is a (nontrivial) p-subgroup of G/H,

$$M(X) = t^X_H\bigl(M(H)\bigr) + \bigl\{x \in M(X)\colon\ r^X_H(x) \in I\bigr\}$$

so that $r^Y_X(y) = t^X_H(a) + b$ for some $a \in M(H)$ and $b \in M(X)$ with $r^X_H(b) \in I$. (Note also that, for $X = H$, such a decomposition of $r^Y_X(y)$ holds trivially, in which $b = 0$.) Now we can write

$$y = t^Y_H(na) + \bigl(y - t^Y_H(na)\bigr).$$

Thus, we may finish the proof by indicating that $r^Y_H(y - t^Y_H(na)) \in I$. Indeed, by using the axioms in the definition of a Mackey functor,

$$\begin{aligned} r^Y_H\big(y - t^Y_H(na)\big) &= r^Y_H(y) - r^Y_H t^Y_H(na) = r^Y_H(y) - \sum_{gX\subseteq Y} c^g_H r^X_H t^X_H(na) \\ &= n\Big(|Y:X| r^Y_H(y) - \sum_{gX\subseteq Y} c^g_H r^X_H t^X_H(a)\Big) \\ &= n\Big(\sum_{gX\subseteq Y} c^g_H r^Y_H(y) - \sum_{gX\subseteq Y} c^g_H r^X_H t^X_H(a)\Big) \\ &= n\sum_{gX\subseteq Y} c^g_H r^X_H\big(r^Y_X(y) - t^X_H(a)\big) = n\sum_{gX\subseteq Y} c^g_H r^X_H(b) \in I, \end{aligned}$$

as desired.

(ii) is equivalent to (iii): Part (2) of 5.15 implies that (iii) is equivalent to the condition $r^X_H(\overline{M}(X) \subseteq r^X_H t^X_H(\overline{M}(H) + \bar{I}$ where X is any subgroup satisfying the required condition. From the Mackey axiom, $r^X_H t^X_H = \sum_{gH\subseteq X} c^g_H$, implying by (the proof of) 4.8 that the above containment relation is equivalent to (ii). □

Remark 5.17. Let A be a finite dimensional algebra, let V be a finite dimensional A-module, and let e be a nonzero idempotent of A.

(1) Let $V_1, V_2, \dots, V_n$ be A-submodules of V. For each i, we put

$$\widetilde{V}_i = \bigcap_{j=1:\ j\neq i}^{n} V_j.$$

For the map

$$\psi : V \to \prod_{i=1}^{n} V/V_i, \qquad v \mapsto \prod_{i=1}^{n} v + V_i,$$

we have:
 (i) If $V_i + \widetilde{V}_i = V$ for each i then ψ is surjective.
 (ii) Suppose further that each V_i is a maximal A-submodule of V. If ψ is surjective then $V_i + \widetilde{V}_i = V$ for each i.

(2) Let $I_1, I_2, \dots, I_n$ be maximal eAe-submodules of eV. Suppose that each $(V :_e I_i)$ is a maximal A-submodule of V. If the product of natural epimorphisms $eV \to \prod_{i=1}^{n} eV/I_i$ is surjective then the product of natural epimorphisms $V \to \prod_{i=1}^{n} V/(V :_e I_i)$ is surjective.

Proof. Left to the reader as an easy exercise. □

Theorem 5.18. *Let $\mathbb{K}$ be of characteristic $p > 0$. Suppose that M is a $\mu_{\mathbb{K}}(G)$-module, H is a subgroup of G, and U is a simple $\mathbb{K}\overline{N}_G(H)$-module. Then, the multiplicity of $S^G_{H,U}$ in $M/\operatorname{Jac}(M)$ is equal to the multiplicity of U in the head of the following quotient module of the $\mathbb{K}\overline{N}_G(H)$-module $\overline{M}(H)$:*

$$\overline{M}(H)/\sum_{X/H}\Big(\bigcap_{f\in\mathcal{A}_X} \operatorname{Ker}\big(f : \overline{M}(H) \to \mathbb{K}\big)\Big)$$

where X/H ranges over all nontrivial p-subgroups of $N_G(H)/H$, and for each X,

$$\mathcal{A}_X = \big\{f \in \operatorname{Hom}_{\mathbb{K}}\big(\overline{M}(H), \mathbb{K}\big):\ f\big(r^X_H t^X_H\big(\overline{M}(H)\big)\big) = 0 \implies f\big(r^X_H\big(\overline{M}(X)\big)\big) = 0\big\}.$$

Proof. Denoting by V_X the space $\bigcap_{f \in \mathcal{A}_X} \operatorname{Ker} f$, let $\mathcal{M}_H = \sum_{X/H} V_X$ where X/H ranges over all nontrivial p-subgroups of $N_G(H)/H$.

We first note that $\mathcal{M}_H$ is a $\mathbb{K}\overline{N}_G(H)$-submodule of $\overline{M}(H)$: As conjugation maps of a Mackey functor are $\mathbb{K}$-space isomorphism, it is clear for any $K \leqslant G$ and $a \in G$ that c_K^a induces a $\mathbb{K}$-space isomorphism $\overline{M}(K) \to \overline{M}({}^gK)$. Moreover, for any $g \in N_G(H)$, it can be seen by the definition of a Mackey functor that $f \in \mathcal{A}_X$ if and only if $f \circ c_H^g \in \mathcal{A}_{X^g}$. So, $c_H^g(V_X) \subseteq V_{{}^gX}$, proving that $\mathcal{M}_H$ is a $\mathbb{K}\overline{N}_G(H)$-module.

As $\overline{M}(H) = \overline{(\downarrow_K^G M)}(H)$ for any $H \leqslant K \leqslant G$, it follows by 3.13 that we may (and will do) assume that H is normal in G. Let n be the multiplicity of $S_{H,U}^G$ in the head of M, and let m be the multiplicity of U in the head of $\overline{M}(H)/\mathcal{M}_H$.

Let $A = \mu_{\mathbb{K}}(G)$, $B = \mathbb{K}\overline{N}_G(H)$, and $e = t_H^H$. There are n maximal A-submodules $J_1, J_2, \ldots, J_n$ of M such that all of the quotients M/J_i are isomorphic to $S_{H,U}^G$ and that the product of natural epimorphisms $M \to \prod_{i=1}^n M/J_i$ is surjective. By 4.6, 4.8 and 5.16 we know that each $J_i(H)/b_H(M)$ is a maximal B-submodule of $\overline{M}(H)$ containing $\mathcal{M}_H$. As the multiplication by the idempotent e is an exact functor (from A-mod to eAe-mod), we see that it induces a surjective eAe-module homomorphism

$$\overline{M}(H) \to \prod_{i=1}^{n} M(H)/J_i(H)$$

with kernel containing $\mathcal{M}_H$. The last surjection induces a surjective B-module homomorphism $\overline{M}(H)/\mathcal{M}_H \to nU$, because B is a unital subalgebra of eAe and each B-module $M(H)/J_i(H)$ is isomorphic to U. This shows that $n \leqslant m$.

Conversely, there are m maximal B-submodules $T_1, T_2, \ldots, T_m$ of $\overline{M}(H)$ containing $\mathcal{M}_H$ such that each B-module $\overline{M}(H)/T_i$ is isomorphic to U and that the product of all natural epimorphisms

$$\overline{M}(H) \to \prod_{i=1}^{m} \overline{M}(H)/T_i$$

is surjective. By the correspondence theorem, there are maximal B-submodules $\overline{I_i} = I_i/b_H(M)$ of $\overline{M}(H)$ such that $T_i = \overline{I_i}/\mathcal{M}_H$. Using the canonical isomorphisms $\overline{M}(H)/T_i \cong M(H)/I_i$, we see that the product of the natural epimorphisms $M(H) \to \prod_{i=1}^m M(H)/I_i$ is surjective. By part (1) of 4.5, each I_i is a maximal eAe-submodule of $eM = M(H)$. Moreover, using 5.16 we see that each $(M :_e I_i)$ is a maximal A-submodule of M. So, we may apply part (2) of 5.17 to deduce that the product of natural epimorphism $M \to \prod_{i=1}^m M/(M :_e I_i)$ is surjective. This shows that $m \leqslant n$, because $M/(M :_e I_i) \cong S_{H,U}^G$ for each i. $\square$

Corollary 5.19. *Let $\mathbb{K}$ be of characteristic $p > 0$. Suppose that M is a $\mu_{\mathbb{K}}(G)$-module, H is a subgroup of G, and U is a simple $\mathbb{K}\overline{N}_G(H)$-module.*

(1) *There is a maximal subfunctor of M whose quotient has H as a minimal subgroup if and only if there is a maximal $\mathbb{K}\overline{N}_G(H)$-submodule $I/b_H(M)$ of $\overline{M}(H)$ satisfying the following condition for any nontrivial p-subgroup X/H of $N_G(H)/H$:*

$$M(X) = t_H^X\big(M(H)\big) + \big\{x \in M(X)\colon r_H^X(x) \in I\big\}.$$

(2) *The multiplicity of $S_{H,U}^G$ in $M/\operatorname{Jac}(M)$ is less than or equal to the multiplicity of U in $\overline{M}(H)/\operatorname{Jac}(\overline{M}(H))$.*

(3) *The multiplicity of $S^G_{H,U}$ in $M/\operatorname{Jac}(M)$ is greater than or equal to the multiplicity of U in the head of the following $\mathbb{K}\overline{N}_G(H)$-module:*

$$\overline{M}(H)/\sum_{H<X\leqslant N_G(H):\ |X:H|=p} r^X_H\big(\overline{M}(X)\big).$$

(4) *Suppose that $\overline{N}_G(H)$ is a p'-group. Then, the multiplicity of $S^G_{H,U}$ in $M/\operatorname{Jac}(M)$ is equal to the multiplicity of U in $\overline{M}(H)$.*

Proof. (1) It follows by (the proof of) 5.18 and 5.16.

(2) and (4) They are immediate from 5.18.

(3) We use the notations $\mathcal{A}_X$, V_X, and $\mathcal{M}_H$ defined in 5.18 and its proof. For any nontrivial p-subgroup X/H of $N_G(H)/H$, part (2) of 5.15 implies that $V_X \subseteq r^X_H(\overline{M}(X))$. Therefore, $\mathcal{M}_H \subseteq \sum_{X/H} r^X_H(\overline{M}(X))$ where X/H ranges over all nontrivial p-subgroup X/H of $N_G(H)/H$. From the transitivity of restriction maps on M, we see that

$$r^X_H\big(M(X)\big) = r^Y_H r^X_Y\big(M(X)\big) \subseteq r^Y_H\big(M(Y)\big), \quad \text{implying that} \quad r^X_H\big(\overline{M}(X)\big) \subseteq r^Y_H\big(\overline{M}(Y)\big)$$

for any subgroup Y/H of X/H of order p. Therefore,

$$\mathcal{M}_H \subseteq \mathcal{N}_H, \quad \text{where } \mathcal{N}_H = \sum_{H<X\leqslant N_G(H):\ |X:H|=p} r^X_H\big(\overline{M}(X)\big).$$

This proves that $\overline{M}(H)/\mathcal{N}_H$ is isomorphic to a quotient module of $\overline{M}(H)/\mathcal{M}_H$. The result now follows from 5.18. $\square$

Proposition 5.20. *Let $\mathbb{K}$ be of characteristic $p > 0$. Suppose that M is a $\mu_{\mathbb{K}}(G)$-module, H is a subgroup of G, and U is a simple $\mathbb{K}\overline{N}_G(H)$-module. If all the elements of order p of $\overline{N}_G(H)$ acts on U trivially, then the multiplicity of $S^G_{H,U}$ in $M/\operatorname{Jac}(M)$ is equal to the multiplicity of U in the head of the following $\mathbb{K}\overline{N}_G(H)$-module*:

$$\overline{M}(H)/\sum_{H<X\leqslant N_G(H):\ |X:H|=p} r^X_H\big(\overline{M}(X)\big).$$

Proof. We use the notations $\mathcal{A}_X$, V_X, $\mathcal{M}_H$ and $\mathcal{N}_H$ defined in 5.18 and 5.19 and their proofs. By 5.18 the multiplicity of $S^G_{H,U}$ in the head of M is equal to the multiplicity of U in the head of $\overline{M}(H)/\mathcal{M}_H$. Let $\varphi : \overline{M}(H) \to U$ be any (nonzero) $\mathbb{K}\overline{N}_G(H)$-module homomorphism whose kernel contains $\mathcal{M}_H$. As $\mathcal{M}_H = \sum_{X/H} V_X$ where X/H ranges over all nontrivial p-subgroups of $N_G(H)/H$, it follows that $V_X \subseteq \operatorname{Ker}\varphi$, in particular, for any subgroup X/H of $N_G(H)/H$ of order p. Using part (2) of 5.15 we see that the containment $V_X \subseteq \operatorname{Ker}\varphi$ is equivalent to the condition

$$r^X_H\big(\overline{M}(X)\big) \subseteq r^X_H t^X_H\big(\overline{M}(H)\big) + \operatorname{Ker}\varphi.$$

We will show that $r^X_H(\overline{M}(X)) \subseteq \operatorname{Ker}\varphi$: It follows by the assumption on U that

$$\varphi\big(r^X_H t^X_H\big(\overline{M}(H)\big)\big) = \varphi\Bigg(\sum_{gH\subseteq X} c^g_H\big(\overline{M}(H)\big)\Bigg) = \sum_{gH\subseteq X} (gH)U = |X:H|U = 0.$$

Therefore, $r^X_H(\overline{M}(X)) \subseteq r^X_H t^X_H(\overline{M}(H)) + \operatorname{Ker}\varphi = \operatorname{Ker}\varphi$. Since this is true for any subgroup X/H of $N_G(H)/H$ of order p, we obtain that $\mathcal{N}_H \subseteq \operatorname{Ker}\varphi$. Consequently, the $\mathbb{K}$-spaces $\operatorname{Hom}_B(\overline{M}(H)/\mathcal{M}_H, U)$ and $\operatorname{Hom}_B(\overline{M}(H)/\mathcal{N}_H, U)$ must be isomorphic where $B = \mathbb{K}\overline{N}_G(H)$. This finishes the proof. $\square$

The comments given after 5.14 are applicable also to the previous result.

6. Some extreme cases

Our aim in this section is to study $\mu_{\mathbb{K}}(G)$-modules M satisfying some extreme conditions such as having a unique maximal or simple subfunctors, and being uniserial.

The result 4.8 contains some necessary and sufficient conditions for a $\mu_{\mathbb{K}}(G)$-module M to have a simple quotient functor of the form $S^G_{H,V}$. It is shown in [11, (15.7) Proposition] that if H is a maximal subgroup of G subject to the condition $\overline{M}(H) \neq 0$ and if we assume that H is normal in G, then for any maximal $\mathbb{K}\overline{N}_G(H)$-submodule $\bar{I}$ of $\overline{M}(H)$, the simple module $V = \overline{M}(H)/\bar{I}$ satisfies the condition (vi) of 4.8 so that M has a simple quotient functor of the form $S^G_{H,V}$. We first want to state this result in a slightly stronger form and then dualize it.

Lemma 6.1. *Let M be a $\mu_{\mathbb{K}}(G)$-module, and let Y and Z be subgroups of G.*

(1) *Assume that $\underline{M}(Y) = 0$. Then, for any $\mathbb{K}$-space homomorphism $\alpha : M(Z) \to M(Y)$,*

$$\operatorname{Ker}\bigl(\alpha : M(Z) \to M(Y)\bigr) = \bigcap_{J<Y} \operatorname{Ker}\bigl(r^Y_J \circ \alpha : M(Z) \to M(J)\bigr).$$

(2) *Assume that $\overline{M}(Y) = 0$. Then, for any $\mathbb{K}$-space homomorphism $\beta : M(Y) \to M(Z)$,*

$$\beta\bigl(M(Y)\bigr) = \sum_{J<Y} \beta \circ t^Y_J\bigl(M(J)\bigr).$$

Proof. We only justify the first part. The second part may be justified similarly. As $\underline{M}(Y) = 0$, it follows by the definition of restriction kernels that the product of restriction maps

$$\varphi = \prod_{J<Y} r^Y_J : M(Y) \to \prod_{J<Y} M(J)$$

is injective. Thus, the kernels of the maps α and $\varphi \circ \alpha$ are equal, implying the result. □

Part (2) of 6.2 can be found in the proof of [11, (15.7) Proposition], whose dual version is part (1) of 6.2.

Lemma 6.2. *Let M be a $\mu_{\mathbb{K}}(G)$-module and H be a subgroup of G.*

(1) *If H is maximal subject to the condition $\underline{M}(H) \neq 0$, then, for any $H < X \leqslant G$,*

$$0 = \bigl(\operatorname{Ker} r^X_H\bigr) \cap \Biggl(\bigcap_{H \not\leqslant J < X} \operatorname{Ker} r^X_J\Biggr).$$

(2) *If H is maximal subject to the condition $\overline{M}(H) \neq 0$, then, for any $H < X \leqslant G$,*

$$M(X) = t^X_H\bigl(M(H)\bigr) + \sum_{H \not\leqslant J < X} t^X_J\bigl(M(J)\bigr).$$

Proof. We only justify the first part by arguing as in the proof of [11, (15.7) Proposition]. Let $H < X \leqslant G$. By the maximality of H,

$$0=\underline{M}(X)=\Bigl(\bigcap_{H\leqslant J<X}\operatorname{Ker}r^X_J\Bigr)\cap\Bigl(\bigcap_{H\nleqslant J<X}\operatorname{Ker}r^X_J\Bigr).$$

For any $H<J<X$, by the maximality of H we obtain that $\underline{M}(J)=0$. Then, part (1) of 6.1 implies (by taking α to be the map r^X_J) that

$$\operatorname{Ker}r^X_J=\bigcap_{K<J}\operatorname{Ker}r^X_K.$$

Substituting this intersection for $\operatorname{Ker}r^X_J$ in the first intersection, and continuing to do this process we finally obtain that

$$0=\bigl(\operatorname{Ker}r^X_H\bigr)\cap\Bigl(\bigcap_{H\nleqslant J<X}\operatorname{Ker}r^X_J\Bigr).\qquad\Box$$

It is proved in [11, (15.7) Proposition] that for a $\mu_{\mathbb{K}}(G)$-module M, a subgroup H of G maximal subject to the condition $\overline{M}(H)\neq 0$, and a simple $\mathbb{K}\overline{N}_G(H)$-module U, the existence of a simple quotient of the functor M isomorphic to $S^G_{H,U}$ is equivalent to the existence of a simple quotient of $\overline{M}(H)$ isomorphic to U. We next show that not only existences but also their multiplicities in respective heads are equal.

Proposition 6.3. *Let M be a $\mu_{\mathbb{K}}(G)$-module, let H be a subgroup of G, and let U be a simple $\mathbb{K}\overline{N}_G(H)$-module.*

(1) *Suppose that H is maximal subject to the condition $\underline{M}(H)\neq 0$. Then, the multiplicity of $S^G_{H,U}$ in* $\operatorname{Soc}(M)$ *is equal to the multiplicity of U in* $\operatorname{Soc}(\underline{M}(H))$.
(2) *Suppose that H is maximal subject to the condition $\overline{M}(H)\neq 0$. Then, the multiplicity of $S^G_{H,U}$ in $M/$*$\operatorname{Jac}(M)$ *is equal to the multiplicity of U in $\overline{M}(H)/$*$\operatorname{Jac}(\overline{M}(H))$.

Proof. (1) Let X/H be a nontrivial p-subgroup of $N_G(H)/H$. We will show that $t^X_H(x)=0$ for any $x\in\underline{M}(H)$ satisfying $(\sum_{gH\subseteq X}c^g_H)x=0$ (that is equivalent to the condition $r^X_Ht^X_H(x)=0$ by the Mackey axiom), from which the result follows by the virtue of 5.12.

Let $x\in\underline{M}(H)$ such that $r^X_Ht^X_H(x)=0$. Then $t^X_H(x)\in\operatorname{Ker}r^X_H$. For any $H\nleqslant J<X$, it follows by the Mackey axiom that

$$r^X_Jt^X_H(x)=\sum_{JgH\subseteq X}t^J_{J\cap H}c^g_{J^g\cap H}r^H_{J^g\cap H}(x).$$

We see that $J^g\cap H\neq H$ for any $g\in N_G(H)$ because $H\nleqslant J$. This shows that $r^H_{J^g\cap H}(x)=0$ because $x\in\underline{M}(H)$. So $r^X_Jt^X_H(x)=0$ for any J with $H\nleqslant J<X$. Consequently,

$$t^X_H(x)\in\bigl(\operatorname{Ker}r^X_H\bigr)\cap\Bigl(\bigcap_{H\nleqslant J<X}\operatorname{Ker}r^X_J\Bigr)=0$$

where the last equality follows from part (1) of 6.2.

(2) Follows from the first part by duality, since $\operatorname{Hom}_{\mu_{\mathbb{K}}(G)}(M,S)\cong\operatorname{Hom}_{\mu_{\mathbb{K}}(G)}(S^*,M^*)$ for any Mackey functors M and S, and since the dual of the simple functor $S^G_{H,V}$ is isomorphic to the simple functor S^G_{H,V^*}. $\Box$

The following is an immediate consequence of 6.3.

Remark 6.4. Let M be a $\mu_{\mathbb{K}}(G)$-module and H be a subgroup of G. If $\underline{M}(H) \neq 0$ (respectively, $\overline{M}(H) \neq 0$), then there is a subgroup K of G containing H such that M has a simple subfunctor (respectively, simple quotient functor) having K as a minimal subgroup.

The next result shows that an example of a $\mu_{\mathbb{K}}(G)$-module M for which H is a maximal subgroup of G subject to the condition $\underline{M}(H) \neq 0$ occurs when $M = \uparrow_H^G T$ for some $\mu_{\mathbb{K}}(H)$-module T such that $\underline{T}(H) \neq 0$.

Proposition 6.5. *Let H be a subgroup of G and T be a $\mu_{\mathbb{K}}(H)$-module. For any subgroup K of G, we have the following $\mathbb{K}\overline{N}_G(K)$-module isomorphisms*:

$$\underline{\left(\uparrow_H^G T\right)}(K) \cong \bigoplus_{N_G(K)gH \subseteq G:\ K^g \leqslant H} \uparrow_{\overline{N}_{{}^gH}(K)}^{\overline{N}_G(K)} {}^g\left(\underline{T}\left(K^g\right)\right),$$

$$\overline{\left(\uparrow_H^G T\right)}(K) \cong \bigoplus_{N_G(K)gH \subseteq G:\ K^g \leqslant H} \uparrow_{\overline{N}_{{}^gH}(K)}^{\overline{N}_G(K)} {}^g\left(\overline{T}\left(K^g\right)\right).$$

In particular, if $\underline{(\uparrow_H^G T)}(K)$ or $\overline{(\uparrow_H^G T)}(K)$ is nonzero, then $K \leqslant_G H$.

Proof. We will prove only the first isomorphism. The second one can be proved similarly. Following the suggestions of the referee we will prove that

$$L^-{}_{N_G(K)/K} \downarrow_{N_G(K)}^G \uparrow_H^G T \cong \bigoplus_{N_G(K)gH \subseteq G:\ K^g \leqslant H} \uparrow_{\overline{N}_{{}^gH}(K)}^{\overline{N}_G(K)} L^-{}_{N_{{}^gH}(K)/K} \downarrow_{N_{{}^gH}(K)}^{{}^gH} |_H^g T$$

as $\mu_{\mathbb{K}}(\overline{N}_G(K))$-modules, from which the first isomorphism is obtained by evaluation at the trivial group K/K.

We begin by justifying two elementary facts. If $L^-{}_{G/N} \uparrow_H^G T$ is nonzero for a normal subgroup N of G then $N \leqslant H$: Indeed, $L^-{}_{G/N} \uparrow_H^G T$, being nonzero, has a simple subfunctor S. Adjointness of the pairs $(\mathrm{Inf}_{G/N}^G, L^-{}_{G/N})$ and $(\downarrow_H^G, \uparrow_H^G)$ imply that $\downarrow_H^G \mathrm{Inf}_{G/N}^G S$ is nonzero, proving that $N \leqslant H$. If N is a normal subgroup of G contained in H then the functors $L^-{}_{G/N} \uparrow_H^G$ and $\uparrow_{H/N}^{G/N} L^-{}_{H/N}$ are naturally isomorphic: Indeed, it is clear from the definitions that the functors $\downarrow_H^G \mathrm{Inf}_{G/N}^G$ and $\mathrm{Inf}_{H/N}^H \downarrow_{H/N}^{G/N}$ are naturally isomorphic. Therefore, their right adjoints $L^-{}_{G/N} \uparrow_H^G$ and $\uparrow_{H/N}^{G/N} L^-{}_{H/N}$ must be naturally isomorphic.

The desired isomorphism is now obtained by using the Mackey decomposition formula (for Mackey functors) and using the facts obtained above. $\square$

Given a subgroup H of G and a $\mu_{\mathbb{K}}(G)$-module M, a smallest element of the set of all subfunctors of M having H as a minimal subgroup may not be a simple functor (but it is indecomposable by the explanation given after 6.7). However, it possesses some properties of simple functors.

Remark 6.6. Let H be a subgroup of G and M be a $\mu_{\mathbb{K}}(G)$-module. Suppose that H is a minimal subgroup of M. Then, M has no proper subfunctor having H as a minimal subgroup if and only if the following conditions hold:

(i) M is generated as a $\mu_{\mathbb{K}}(G)$-module by its value $M(H)$.
(ii) H is the unique, up to G-conjugacy, minimal subgroup of M.
(iii) $M(H)$ is a simple $\mathbb{K}\overline{N}_G(H)$-module.

Proof. For any $\mu_{\mathbb{K}}(G)$-module M, it is clear by the definition of restriction kernels that if H is a minimal subgroup of M then $\underline{M}(H) = M(H) \neq 0$. The result follows by part (3) of 5.3 and by the bijective correspondence given in 5.4. $\square$

Any simple $\mu_{\mathbb{K}}(G)$-module M having H as a minimal subgroup satisfies the conditions (i)–(iii) of the previous result so that the previous result explains what happens in the converse situation of [10, (2.3) Proposition].

Proposition 6.7. *Let M be a $\mu_{\mathbb{K}}(G)$-module and H be a subgroup of G. Put $A = \mu_{\mathbb{K}}(G)$ and $e = t_H^H$. Then:*

(1) *For any $\mathbb{K}\overline{N}_G(H)$-submodule T of $\underline{M}(H)$, the maps $J \to J(H)$ and $(AT :_e I) \leftarrow I$ define a bijective correspondence between the maximal $\mu_{\mathbb{K}}(G)$-submodules J of AT and the maximal $\mathbb{K}\overline{N}_G(H)$-submodules I of T. Moreover, any simple quotient functor of AT has H as a minimal subgroup.*

(2) *For any $\mathbb{K}\overline{N}_G(H)$-submodule $\overline{I}$ of $\overline{M}(H)$ where $\overline{I} = I/b_H(M)$, the maps $S \to S(H)$ and $AT \leftarrow T$ define a bijective correspondence between the simple $\mu_{\mathbb{K}}(G)$-submodules S of $\widetilde{M} = M/(M :_e I)$ and the simple $\mathbb{K}\overline{N}_G(H)$-submodules T of $\widetilde{M}(H) \cong \overline{M}(H)/\overline{I}$. Moreover, any simple subfunctor of $\widetilde{M}$ has H as a minimal subgroup.*

Proof. Put $B = \mathbb{K}\overline{N}_G(H)$.

(1) Part (1) of 5.3 implies that eAe-submodules and B-submodules of T are the same. As $eAT = T$ and $AT = Ae(AT)$, the required bijection follows from 3.7. It follows by this bijection that any simple quotient of AT is of the form $AT/(AT :_e I)$ for some maximal B-submodule I of T. The value of $AT/(AT :_e I)$ at H is isomorphic to T/I which is nonzero. For any $X < H$, part (3) of 5.3 implies that $(AT)(X) = 0$. Consequently, H is a minimal subgroup of $AT/(AT :_e I)$.

(2) Firstly, using part (5) of 4.1 we see that $(\widetilde{M} :_e 0) = 0$. Moreover, it follows by part (2) of 4.5 that $\widetilde{M}(X) = 0$ for all $X < H$. Now, by using 3.3 and part (1) of 4.5, and by arguing as in the first part, one may prove the results. $\square$

Let M a $\mu_{\mathbb{K}}(G)$-module and H be a subgroup of G. Let M' be any smallest element of the set of all subfunctors of M having H as a minimal subgroup. It follows by 5.4 that $M' = AT$ for some simple $\mathbb{K}\overline{N}_G(H)$-submodule T of $\underline{M}(H)$, where $A = \mu_{\mathbb{K}}(G)$. We see by using part (1) of 6.7 that M' has a unique maximal subfunctor implying that M' is indecomposable. In particular, any $\mu_{\mathbb{K}}(G)$-module satisfying the conditions of 6.6 has a unique maximal subfunctor (and so it is indecomposable).

Let M a $\mu_{\mathbb{K}}(G)$-module and H be a subgroup of G. Given any composition series

$$0 = T_0 \subset T_1 \subset \cdots \subset T_{n-1} \subset T_n = \underline{M}(H)$$

of the $\mathbb{K}\overline{N}_G(H)$-module $\underline{M}(H)$, letting $A = \mu_{\mathbb{K}}(G)$ we obtain the series

$$0 = AT_0 \subset AT_1 \subset \cdots \subset AT_{n-1} \subset AT_n = A\underline{M}(H)$$

of $\mu_{\mathbb{K}}(G)$-submodules of M. The inclusions $AT_{i-1} \subseteq AT_i$ are strict because $eAT_i = T_i$ where $e = t_H^H$. Part (1) of 6.7 implies that $(AT_i :_e T_{i-1})$ is a maximal $\mu_{\mathbb{K}}(G)$-submodule of AT_i whose quotient $AT_i/(AT_i :_e T_{i-1})$ is isomorphic to S_{H,V_i}^G, where V_i is isomorphic to T_i/T_{i-1}. Moreover, we see by part (1) of 4.1 that $AT_{i-1} \subseteq (AT_i :_e T_{i-1})$. Consequently, we have proved for any simple $\mathbb{K}\overline{N}_G(H)$-module V that the multiplicity of the simple $\mu_{\mathbb{K}}(G)$-module $S_{H,V}^G$ as a composition factor of M (indeed, of $A\underline{M}(H)$) is greater than or equal to the multiplicity of V as a composition factor of $\underline{M}(H)$. This is the dual version of [11, (6.2) Proposition]. Moreover, as the evaluation of $(AT_i :_e T_{i-1})/AT_{i-1}$ at subgroups of H are 0, we see that the multiplicity of V as a composition factor of $\underline{M}(H)$ is equal to the multiplicity of $S_{H,V}^G$ as a composition factor of $A\underline{M}(H)$. This can also be deduced by using the next result.

Proposition 6.8. *Let M be a $\mu_{\mathbb{K}}(G)$-module and H be a minimal subgroup of M. Then, for any simple $\mathbb{K}\overline{N}_G(H)$-module V, the multiplicity of $S^G_{H,V}$ as a composition factor of M is equal to the multiplicity of V as a composition factor of $M(H)$.*

Proof. Let $0 = M_0 \subset M_1 \subset \cdots \subset M_{n-1} \subset M_n = M$ be a composition series of M. Evaluating at H yields a series

$$0 = M_0(H) \subseteq M_1(H) \subseteq \cdots \subseteq M_{n-1}(H) \subseteq M_n(H) = M(H)$$

of $\mathbb{K}\overline{N}_G(H)$-submodules of $M(H)$. Each M_i/M_{i-1} is isomorphic to a simple $\mu_{\mathbb{K}}(G)$-module of the form $S_i = S^G_{H_i,V_i}$ for some H_i and V_i. We will show that $M_{i-1}(H) \neq M_i(H)$ if and only if $H_i =_G H$. This clearly finishes the proof, because the isomorphism of two simple functors of the form $S^G_{A,U}$ and $S^G_{B,W}$ is equivalent to the existence of a $g \in G$ satisfying $B = {}^gA$ and $W \cong {}^gU$ and because $S^G_{A,U}(A) \cong U$ for any simple functor $S^G_{A,U}$ (see 2.2).

As $S_i(H_i) \neq 0$, we see that $M_{i-1}(H_i) \neq M_i(H_i)$ and that $M_i(H_i) \neq 0$. From $0 \neq M_i(H_i) \subseteq M(H_i)$ we obtain that $H_i \not<_G H$ because H is a minimal subgroup of M. On the other hand, if $M_{i-1}(H) \neq M_i(H)$ then $S_i(H) \neq 0$ implying that $H_i \leqslant_G H$. Consequently, $M_{i-1}(H) \neq M_i(H)$ if and only if $H_i =_G H$. $\square$

It is clear that a $\mu_{\mathbb{K}}(G)$-module M has a composition factor having 1 as a minimal subgroup if and only if $M(1) \neq 0$. Therefore, taking $H = 1$ in 6.8 one obtains [11, (6.3) Proposition].

Corollary 6.9. *Let M be a $\mu_{\mathbb{K}}(G)$-module, let H be a subgroup of G, and let V be a simple $\mathbb{K}\overline{N}_G(H)$-module. Put $A = \mu_{\mathbb{K}}(G)$ and $e = t^H_H$. Then:*

(1) *For any $\mathbb{K}\overline{N}_G(H)$-submodule T of $\underline{M}(H)$, the multiplicity of $S^G_{H,V}$ as a composition factor of AT is equal to the multiplicity of V as a composition factor of T.*
(2) *For any $\mathbb{K}\overline{N}_G(H)$-submodule $\overline{I} = I/b_H(M)$ of $\overline{M}(H)$, the multiplicity of $S^G_{H,V}$ as a composition factor of $M/(M :_e I)$ is equal to the multiplicity of V as a composition factor of $\overline{M}(H)/\overline{I}$.*

Proof. (1) Using part (3) of 5.3 we see that H is a minimal subgroup of the functor AT. Then, the result follows from 6.8, because $(AT)(H) = T$.

(2) Using part (2) of 4.5 we see that H is a minimal subgroup of the functor $M/(M :_e I)$. Then, the result follows from 6.8, because the evaluation of $M/(M :_e I)$ at H is isomorphic to $\overline{M}(H)/\overline{I}$. $\square$

Now we can see the precise version of the situation about multiplicities explained at the beginning of 6.8. For a $\mu_{\mathbb{K}}(G)$-module M, a subgroup H of G, and a simple $\mathbb{K}\overline{N}_G(H)$-module V, part (1) of 6.9 implies that the multiplicity of V as a composition factor of $\underline{M}(H)$ is equal to the multiplicity of $S^G_{H,V}$ as a composition factor of $A\underline{M}(H)$ where $A = \mu_{\mathbb{K}}(G)$. In a similar way, the multiplicity of V as a composition factor of $\overline{M}(H)$ is equal to the multiplicity of $S^G_{H,V}$ as a composition factor of $M/(M :_e b_H(M))$ where $e = t^H_H$.

If a $\mu_{\mathbb{K}}(G)$-module M has a unique maximal submodule whose simple quotient has H as a minimal subgroup, then it follows by 6.3 that H is the unique, up to G-conjugacy, maximal subgroup of G subject to the condition $\overline{M}(H) \neq 0$. We next want to study such $\mu_{\mathbb{K}}(G)$-modules including the uniserial ones. A finite dimensional module of an algebra is said to be uniserial if its submodule lattice is a chain, equivalently if it has a unique composition series.

Lemma 6.10. *Let M be a $\mu_{\mathbb{K}}(G)$-module, and let H and K be subgroups of G. Put $A = \mu_{\mathbb{K}}(G)$.*

(1) *Suppose that $\overline{M}(H) \neq 0$. If $AM(H) \subseteq AM(K)$ then $H \leqslant_G K$.*
(2) *Suppose that $\underline{M}(H) \neq 0$. If $A\underline{M}(H) \subseteq A\underline{M}(K)$ then $K \leqslant_G H$.*

Proof. (1) Evaluation at H gives that $M(H) \subseteq t_H^H \mu_{\mathbb{K}}(G) t_K^K M(K)$. Using the basis Theorem 2.1 we see that

$$M(H) \subseteq t_H^H \mu_{\mathbb{K}}(G) t_K^K M(K) = \sum_{g \in G,\, J \leqslant H^g \cap K} t_{{}^g J}^H c_J^g r_J^K \big(M(K)\big).$$

If ${}^g J < H$ for any g and J appearing in the above sum, then the sum is in $b_H(M)$ so that $M(H) \subseteq b_H(M)$ contradicting the assumption $\overline{M}(H) \neq 0$. So there is a $g \in G$ and $J \leqslant H^g \cap K$ satisfying ${}^g J = H$. This shows that $H \leqslant_G K$.

(2) We obtain by evaluation at H that $0 \neq \underline{M}(H) \subseteq t_H^H \mu_{\mathbb{K}}(G) t_K^K \underline{M}(K)$. As $r_J^K(\underline{M}(K)) = 0$ for any $J < K$, arguing as in the first part we see by using the basis Theorem 2.1 that $J = K$ for some $g \in G$ and $J \leqslant H^g \cap K$. This shows that $K \leqslant_G H$. $\square$

In the next result we observe that the primordial subgroups of a uniserial $\mu_{\mathbb{K}}(G)$-module M (i.e., subgroups X of G for which $\overline{M}(X) \neq 0$) form a chain with respect to the subgroup conjugacy relation $\leqslant_G$.

Proposition 6.11. *Let M be a uniserial $\mu_{\mathbb{K}}(G)$-module, and let H and K be subgroups of G.*

(1) *If $\overline{M}(H) \neq 0$ and $\overline{M}(K) \neq 0$, then $H \leqslant_G K$ or $K \leqslant_G H$.*
(2) *If $\underline{M}(H) \neq 0$ and $\underline{M}(K) \neq 0$, then $H \leqslant_G K$ or $K \leqslant_G H$.*

Proof. As the justifications of both parts are similar, we only justify the first part. Since M is uniserial, we must have that $AM(H) \subseteq AM(K)$ or $AM(K) \subseteq AM(H)$ where $A = \mu_{\mathbb{K}}(G)$. Part (1) of 6.10 implies that $H \leqslant_G K$ or $K \leqslant_G H$. $\square$

Lemma 6.12. *Let M be a $\mu_{\mathbb{K}}(G)$-module for which there is a unique, up to G-conjugacy, subgroup H of G maximal subject to the condition $\overline{M}(H) \neq 0$. If $M_2 \subseteq M_1$ are $\mu_{\mathbb{K}}(G)$-submodules of M such that $M/M_1 \cong S_{H,V}^G$ and $M_1/M_2 \cong S_{K,W}^G$ for some simple $\mu_{\mathbb{K}}(G)$-modules $S_{H,V}^G$ and $S_{K,W}^G$, then $H \leqslant_G K$ or $K \leqslant_G H$.*

Proof. Assume that $H \nleqslant_G K$. Take any $X \leqslant K$. Then $H \nleqslant_G X$. Evaluation of $M/M_1 \cong S_{H,V}^G$ at X is 0 implying that $M(X) = M_1(X)$. Thus, $b_K(M) = b_K(M_1)$ and $M(K) = M_1(K)$ so that $\overline{M}(K) = \overline{M_1}(K)$. As $M_1/M_2 \cong S_{K,W}^G$, it follows from 4.4 that $\overline{M_1}(K) \neq 0$. Hence $\overline{M}(K) \neq 0$, and the maximality of H implies that $K \leqslant_G H$. $\square$

We now observe that the minimal subgroups of any two successive simple factors of the composition series of a uniserial $\mu_{\mathbb{K}}(G)$-module can be compared with respect to the subgroup conjugacy relation $\leqslant_G$.

Proposition 6.13. *Let M be a uniserial $\mu_{\mathbb{K}}(G)$-module with the composition series*

$$0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_{n-1} \subset M_n = M \quad \text{where } M_i/M_{i-1} \cong S_{H_i,V_i}^G,$$

for each i. Then, $H_i \leqslant_G H_{i-1}$ or $H_{i-1} \leqslant_G H_i$ for each i.

Proof. The $\mu_{\mathbb{K}}(G)$-module M_i is uniserial for each i. In particular, M_{i-1} is the unique maximal $\mu_{\mathbb{K}}(G)$-submodule of M_i. So, 6.3 implies that H_i is the unique, up to G-conjugacy, maximal subgroup of G subject to the condition $\overline{M}(H_i) \neq 0$. Now the result follows from 6.12 applied to the submodules $M_{i-2} \subseteq M_{i-1}$ of M_i. $\square$

The previous result may also be deduced as an immediate consequence of [11, (14.3) Theorem] involving a condition for Ext groups of simple functors to be 0. Indeed, in the case of 6.13, one has a non-split exact sequence $0 \to S^G_{H_{i-1},V_{i-1}} \to M_i/M_{i-2} \to S^G_{H_i,V_i} \to 0$ so that $\mathrm{Ext}^1_{\mu_{\mathbb{K}}(G)}(S^G_{H_i,V_i}, S^G_{H_{i-1},V_{i-1}}) \neq 0$, implying by the above mentioned result of [11] that $H_i \leqslant_G H_{i-1}$ or $H_{i-1} \leqslant_G H_i$. Moreover, by using [11, (14.6) Theorem] one conclude more that $H_i \trianglelefteq {}^g H_{i-1}$ or $H_{i-1} \trianglelefteq {}^g H_i$ for some $g \in G$.

Proposition 6.14. *Let $\mathbb{K}$ be of characteristic $p > 0$. Let M be a $\mu_{\mathbb{K}}(G)$-module having a unique maximal $\mu_{\mathbb{K}}(G)$-submodule, say $M/\mathrm{Jac}(M) \cong S^G_{H,V}$, and let K be a subgroup of G such that $\overline{M}(K) \neq 0$, and let X be a subgroup of G such that $S^G_{H,V}(X) \neq 0$. Then*:

1. *$\overline{M}(H)$ has a unique maximal $\mathbb{K}\overline{N}_G(H)$-submodule, and the simple head of the $\mathbb{K}\overline{N}_G(H)$-module $\overline{M}(H)$ is isomorphic to V.*
2. *$K \leqslant_G H$, and if $K \neq_G H$ then p divides $|N_G(K) : K|$.*
3. *M is generated as a $\mu_{\mathbb{K}}(G)$-module by its value $M(X)$ at X.*

Proof. (1) Put $J = \mathrm{Jac}(M)$. We see that J is the unique largest element of the set of all subfunctors J' of M whose quotient M/J' has H as a minimal subgroup. Then 4.4 implies that $\overline{M}(H)$ has a unique maximal $\mathbb{K}\overline{N}_G(H)$-submodule, which is $J(H) = \mathrm{Jac}(\overline{M}(H))$. Moreover, evaluating the isomorphic functors M/J and $S^G_{H,V}$ at H we see that the head of $\overline{M}(H)$ is isomorphic to V.

(2) Choose a maximal subgroup L of G containing K subject to the condition $\overline{M}(L) \neq 0$. It follows from 6.3 that M has a maximal $\mu_{\mathbb{K}}(G)$-submodule whose simple quotient has L as a minimal subgroup. As M has a unique maximal $\mu_{\mathbb{K}}(G)$-submodule, $L =_G H$ so that $K \leqslant_G H$. Moreover, if $K \neq_G H$ then part (4) of 5.19 implies that p divides $\overline{N}_G(K)$.

(3) Put $J = \mathrm{Jac}(M)$, $A = \mu_{\mathbb{K}}(G)$ and $e = t^X_X$. The idempotent $e \in A$ does not annihilate the simple A-module M/J. Then part (7) of 4.1 implies that $AeM + J = M$. If $AeM \neq M$ then, as J contains every proper A-submodule of M, it follows that $J = M$, which is not the case. Hence $AeM = M$. □

The following dual version of the previous result may be justified similarly.

Proposition 6.15. *Let $\mathbb{K}$ be of characteristic $p > 0$. Let M be a $\mu_{\mathbb{K}}(G)$-module having a unique simple $\mu_{\mathbb{K}}(G)$-submodule, say $\mathrm{Soc}(M) \cong S^G_{H,V}$, and let K be a subgroup of G such that $\underline{M}(K) \neq 0$, and let X be a subgroup of G such that $S^G_{H,V}(X) \neq 0$. Then*:

1. *$\underline{M}(H)$ has a unique simple $\mathbb{K}\overline{N}_G(H)$-submodule, and the simple socle of the $\mathbb{K}\overline{N}_G(H)$-module $\underline{M}(H)$ is isomorphic to V.*
2. *$K \leqslant_G H$, and if $K \neq_G H$ then p divides $|N_G(K) : K|$.*
3. *$(M :_e 0) = 0$ where $e = t^X_X$.*

Let A be a finite dimensional $\mathbb{K}$-algebra and V be an A-module. If V is isomorphic to a nonzero quotient module of a projective indecomposable A-module P then it is clear that the heads of P and V are isomorphic so that the head of V is a simple A-module. Conversely, if the head of V is isomorphic to a simple A-module S then there are A-module epimorphisms $\pi : V \to S$ and $f : P(S) \to S$ where $P(S)$ is the projective cover of S. By the projectivity of $P(S)$ we may find an A-module homomorphisms $\gamma : P(S) \to V$ satisfying $\pi \circ \gamma = f$. Using the relation $\pi \circ \gamma = f$ one sees that $\gamma : P(S) \to V$ is an epimorphism. Hence, an A-module has unique maximal submodule if and only if it is isomorphic to a nonzero quotient of a projective indecomposable A-module. In a similar way, one sees that a module has unique simple submodule if and only if it is isomorphic to a submodule of an injective indecomposable module.

As in [11] we denote by $P^G_{H,V}$ the projective cover of a simple $\mu_{\mathbb{K}}(G)$-module of the form $S^G_{H,V}$. Thus, 6.14 applies to $P^G_{H,V}$ and its nonzero quotients.

Remark 6.16. Let M be a uniserial $\mu_{\mathbb{K}}(G)$-module. Then, for any subgroup H of G, the $\mathbb{K}\overline{N}_G(H)$-modules $\underline{M}(H)$ and $\overline{M}(H)$ are uniserial.

Proof. Let T_1 and T_2 be $\mathbb{K}\overline{N}_G(H)$-submodules of $\underline{M}(H)$. By part (1) of 5.3 they are also eAe-submodules of $M(H)$ where $A=\mu_{\mathbb{K}}(G)$ and $e=t_H^H$. Therefore, $eAT_i=T_i$ for each i. As M is a uniserial A-module, its A-submodules AT_1 and AT_2 must be comparable, say $AT_1\subseteq AT_2$. Multiplying this containment by the idempotent e we get $T_1\subseteq T_2$. Hence, $\underline{M}(H)$ is uniserial. Similar arguments may be used to justify the result for $\overline{M}(H)$. $\square$

As an easy consequence of 6.14 and 3.7 we obtain the following criterion for a $\mu_{\mathbb{K}}(G)$-module to have a unique maximal submodule.

Remark 6.17. Let M be a $\mu_{\mathbb{K}}(G)$-module. Then, M has a unique maximal $\mu_{\mathbb{K}}(G)$-submodule if and only if there is a subgroup H of G satisfying the following conditions:

(i) M is generated as a $\mu_{\mathbb{K}}(G)$-module by its value $M(H)$ at H.
(ii) $M(H)$ has a unique maximal $t_H^H\mu_{\mathbb{K}}(G)t_H^H$-submodule.

It is desirable to replace the second condition of 6.17 with a condition involving $\overline{M}(H)$ and $\mathbb{K}\overline{N}_G(H)$. This can be done if $\mathbb{K}$ is of characteristic $p>0$ and G is a p-group, because in this case it follows from [11, (15.1) Lemma] that $S^G_{K,\mathbb{K}}(X)\neq 0$ implies $X=_G K$.

The next result is a slightly more general form of [11, (15.1) Lemma].

Lemma 6.18. *Let $\mathbb{K}$ be of characteristic $p>0$ and $S^G_{H,V}$ be a simple $\mu_{\mathbb{K}}(G)$-module. Let K be a subgroup of G. Assume that either K is a normal subgroup of G or $\dim_{\mathbb{K}} V=1$. Then, $S^G_{H,V}(K)\neq 0$ if and only if there is a $g\in G$ with ${}^gH\leqslant K$ satisfying the following conditions*:

(i) *$\overline{N}_K({}^gH)$ acts on ${}^gV$ trivially.*
(ii) *p does not divide $|N_K({}^gH):{}^gH|$.*

Proof. We first try to find conditions equivalent to the condition $S^G_{H,V}(G)\neq 0$: Using the isomorphism given in 2.7 and using the explicit description of induced functors given in 2.3 we see that

$$S^G_{H,V}(G)\cong S^{\overline{N}_G(H)}_{1,V}\big(\overline{N}_G(H)\big)=\mathrm{tr}_1^{\overline{N}_G(H)}(V)\subseteq V^{\overline{N}_G(H)}$$

where tr denotes the relative trace map, because $S^{\overline{N}_G(H)}_{1,V}$ is the (unique simple) subfunctor of the fixed point functor $FP_V^{\overline{N}_G(H)}$ generated by $FP_V^{\overline{N}_G(H)}(1)=V$, see [10] for more details about the fixed point functors. Note that $V^{\overline{N}_G(H)}$ is a submodule of the simple $\mathbb{K}\overline{N}_G(H)$-module V. Thus, if $S^G_{H,V}(G)\neq 0$ then $V=V^{\overline{N}_G(H)}$ implying that $\overline{N}_G(H)$ acts on V trivially (i.e., V is the trivial module). Moreover, if V is the trivial module then we see that $S^G_{H,V}(G)\cong|N_G(H):H|V$. Consequently, $S^G_{H,V}(G)\neq 0$ if and only if $\overline{N}_G(H)$ acts on V trivially and p does not divide $|N_G(H):H|$.

Let K and V satisfy the conditions of the hypothesis. If K is normal or if $\dim_{\mathbb{K}} V=1$, then Clifford's theorem for Mackey algebras [13] or 3.17 implies respectively that $\downarrow^G_K S^G_{H,V}$ is semisimple. Thus, $0\neq S^G_{H,V}(K)=(\downarrow^G_K S^G_{H,V})(K)$ if and only if there is a simple $\mu_{\mathbb{K}}(K)$-module S direct summand of the semisimple $\mu_{\mathbb{K}}(K)$-module $\downarrow^G_K S^G_{H,V}$ such that $S(K)\neq 0$. It follows by 3.15 that simple direct summands of the semisimple $\mu_{\mathbb{K}}(K)$-module $\downarrow^G_K S^G_{H,V}$ are precisely of the form $S^K_{{}^gH,W}$ where $g\in G$ with ${}^gH\leqslant K$ and W is a simple $\mathbb{K}\overline{N}_K({}^gH)$-submodule of ${}^gV$. Thus, $S^G_{H,V}(K)\neq 0$ if and only if $S^K_{{}^gH,W}(K)\neq 0$ for some $g\in G$ with ${}^gH\leqslant K$ and for some simple $\mathbb{K}\overline{N}_K({}^gH)$-submodule W of ${}^gV$. This is, by what we have proved in the first paragraph, equivalent to the requirements that W is the trivial $\mathbb{K}\overline{N}_K({}^gH)$-module and that p does not divide $|N_K({}^gH):{}^gH|$. If $\dim_{\mathbb{K}} V=1$ then $W={}^gV$ so that the result follows.

Assume that K is normal in G. Then $N_K({}^gH)$ is normal in $N_G({}^gH)$. Take any simple $\mathbb{K}\overline{N}_K({}^gH)$-submodule U of ${}^gV$. Then Clifford's theorem for group algebras implies that any simple direct summand of the semisimple $\mathbb{K}\overline{N}_K({}^gH)$-module ${}^gV$ is an $\overline{N}_G({}^gH)$-conjugate of U. Therefore, $\mathbb{K}\overline{N}_K({}^gH)$ acts on U trivially if and only if it acts on ${}^gV$ trivially. □

Proposition 6.19. *Let $\mathbb{K}$ be of characteristic $p > 0$ and G be a p-group. Let M be a $\mu_{\mathbb{K}}(G)$-module. Then*:

(1) *M has a unique simple $\mu_{\mathbb{K}}(G)$-submodule if and only if there is a subgroup H of G satisfying the following conditions*:
  (i) *$(M :_e 0) = 0$ where $e = t_H^H$.*
  (ii) *$\underline{M}(H)$ has a unique simple $\mathbb{K}\overline{N}_G(H)$-submodule.*
(2) *M has a unique maximal $\mu_{\mathbb{K}}(G)$-submodule if and only if there is a subgroup H of G satisfying the following conditions*:
  (i) *M is generated as a $\mu_{\mathbb{K}}(G)$-module by its value $M(H)$ at H.*
  (ii) *$\overline{M}(H)$ has a unique maximal $\mathbb{K}\overline{N}_G(H)$-submodule.*

Proof. We only prove the first part. If M has a unique simple subfunctor, say of the form $S^G_{H,\mathbb{K}}$, then it follows from 6.15 that the subgroup H satisfies the desired conditions. Suppose that there is a subgroup H of G satisfying the given conditions. It follows from $(M :_e 0) = 0$ that M has no nonzero subfunctor whose evaluation at H is 0. Thus, if M has a simple subfunctor of the form $S^G_{K,\mathbb{K}}$ then $S^G_{K,\mathbb{K}}(H) \neq 0$ implying by 6.18 that $K =_G H$. Consequently, any simple subfunctor of M has H as a minimal subgroup. Now 5.4 implies that M has a unique simple subfunctor. □

Proposition 6.20. *Let $\mathbb{K}$ be of characteristic $p > 0$ and G be a p-group. Let H be a subgroup of G, and let M be a $\mu_{\mathbb{K}}(G)$-module. Put $e = t_H^H$ and*

$$b_H^0(M) = \sum_{H<K:\ |K:H|=p} r_H^K\big(M(K)\big) + b_H(M),$$

which is a $\mathbb{K}\overline{N}_G(H)$-module. Then, the maps $J \to J(H)$ and $(M :_e I) \leftarrow I$ define a bijective correspondence between the maximal $\mu_{\mathbb{K}}(G)$-submodules J of M such that $M/J \cong S^G_{H,\mathbb{K}}$ and the maximal $\mathbb{K}\overline{N}_G(H)$-submodules I of $M(H)$ containing $b_H^0(M)$. Moreover, $b_H^0(M) \subseteq \mathrm{Jac}(M)(H)$ and the $\mathbb{K}\overline{N}_G(H)$-module $\mathrm{Jac}(M)(H)/b_H^0(M)$ is the radical of $M(H)/b_H^0(M)$.

Proof. Let J be a subfunctor of M such that $M/J \cong S^G_{H,\mathbb{K}}$. For any $K > H$, it follows from 6.18 that r_H^K annihilates M/J so that $r_H^K(M(K)) \subseteq J(H)$. We also know from 4.4 that $b_H(M) \subseteq J(H)$. Therefore, $J(H)$ contains $b_H^0(M)$.

Let I be a $\mathbb{K}\overline{N}_G(H)$-submodule of $M(H)$ containing $b_H^0(M)$. Take any $X > H$. By the transitivity of restriction maps (i.e., $r_A^B r_B^C = r_A^C$ for $A \leqslant B \leqslant C$) we see that $r_H^K(M(K)) \subseteq I$ for any $K > H$. Therefore, $\{x \in M(X)\colon\ c^g_{H^g} r^X_{H^g}(x) \in I,\ \ \forall g \in G,\ \ H^g \leqslant X\} = M(X)$ so that we can deduce the maximality of the subfunctor $(M :_e I)$ from part (1) of 4.6. Now, the required bijection follows from 4.4.

For any maximal subfunctor J' of M with $M/J' \cong S^G_{K,\mathbb{K}}$, if $K \neq_G H$ then 6.18 implies that $J'(H) = M(H)$. Thus, $\mathrm{Jac}(M)(H)$ is the intersection of all $J(H)$ where J ranges over all maximal subfunctors of M with $M/J \cong S^G_{H,\mathbb{K}}$. By the bijective correspondence proved above, we see that $b_H^0(M) \subseteq \mathrm{Jac}(M)(H)$ and the quotient is the radical of $M(H)/b_H^0(M)$. □

Regarding simple subfunctors, one may prove the following similar to 6.20.

Proposition 6.21. *Let $\mathbb{K}$ be of characteristic $p > 0$ and G be a p-group. Let H be a subgroup of G, and let M be a $\mu_{\mathbb{K}}(G)$-module. Put $A = \mu_{\mathbb{K}}(G)$ and*

$$k_H^0(M) = \bigcap_{H<K:\ |K:H|=p} \operatorname{Ker}\bigl(t_H^K : \underline{M}(H) \to M(K)\bigr),$$

which is a $\mathbb{K}\overline{N}_G(H)$-module. Then, the maps $S \to S(H)$ and $AT \leftarrow T$ define a bijective correspondence between the simple $\mu_{\mathbb{K}}(G)$-submodules S of M such that $S \cong S_{H,\mathbb{K}}^G$ and the simple $\mathbb{K}\overline{N}_G(H)$-submodules T of $M(H)$ contained in $k_H^0(M)$. Moreover, $\operatorname{Soc}(M)(H) \subseteq k_H^0(M)$ and the $\mathbb{K}\overline{N}_G(H)$-module $\operatorname{Soc}(M)(H)$ is the socle of $k_H^0(M)$.

Let V be a finite dimensional module over an algebra. For any natural number $i \geqslant 1$ we put $\operatorname{Jac}^i(V) = \operatorname{Jac}(\operatorname{Jac}^{i-1}(V))$ and $\operatorname{Soc}^i(V)/\operatorname{Soc}^{i-1}(V) = \operatorname{Soc}(V/\operatorname{Soc}^{i-1}(V))$ where $\operatorname{Jac}^0(V) = V$ and $\operatorname{Soc}^0(V) = 0$. One has the radical series $V = \operatorname{Jac}^0(V) \supset \operatorname{Jac}^1(V) \supset \cdots \supset \operatorname{Jac}^n(V) = 0$ of V, and the socle series $0 = \operatorname{Soc}^0(V) \subset \operatorname{Soc}^1(V) \subset \cdots \subset \operatorname{Soc}^m(V) = V$ of V. The lengths of the radical series and the socle series of V are equal (i.e., $n = m$), and it is called the Loewy length of V.

We next state a result giving a lower bound for Loewy lengths.

Proposition 6.22. *Let $\mathbb{K}$ be of characteristic $p > 0$ and G be a p-group. Let M be a $\mu_{\mathbb{K}}(G)$-module, and $H \leqslant K$ be subgroups of G with $|K : H| = p^n$. If $t_H^K(M(H)) \neq 0$ or $r_H^K(M(K)) \neq 0$, then the Loewy length of M is greater than or equal to $n+1$.*

Proof. For any natural number k let $J_k = \operatorname{Jac}^k(M)$. If $X \leqslant Y$ are subgroups of G with $|Y : X| = p$, then it follows by 6.18 that both of the elements t_X^Y and r_X^Y of $\mu_{\mathbb{K}}(G)$ annihilate the semisimple $\mu_{\mathbb{K}}(G)$-modules, in particular J_k/J_{k+1}. This gives that $t_X^Y(J_k(X)) \subseteq J_{k+1}(Y)$ and $r_X^Y(J_k(Y)) \subseteq J_{k+1}(X)$. Using the transitivity of trace and restriction maps on a Mackey functor, the above argument can be used repeatedly to obtain that $t_A^B(J_k(A)) \subseteq J_{k+m}(B)$ and $r_A^B(J_k(B)) \subseteq J_{k+m}(A)$ where $|B : A| = p^m$. Therefore, $0 \neq t_H^K(M(H)) = t_H^K(J_0(H)) \subseteq J_n(H)$. This shows that the Loewy length of M is at least $n+1$. $\square$

If $\mathbb{K}$ is of characteristic $p > 0$ and G is a p-group, then one may see that the Loewy length of the fixed point functor $FP_{\mathbb{K}}^G$ is $n+1$ where $|G| = p^n$, see 7.20. As the restriction maps of $FP_{\mathbb{K}}^G$ are all injective, $r_1^G(FP_{\mathbb{K}}^G) \neq 0$ so that the lower bound obtained by 6.22 is attained by the Loewy length of $FP_{\mathbb{K}}^G$.

7. The Burnside functor

In this section we want to study the radical and the socle series of the Burnside functor $\mathrm{B}_{\mathbb{K}}^G$ for G over $\mathbb{K}$. We begin with recalling the maps between Burnside algebras of subgroups of G making $\mathrm{B}_{\mathbb{K}}^G$ a Mackey functor for G, see [1,2,11]. Let H be a subgroup of G. The set of isomorphism classes of finite H-sets form a commutative semiring under the operations disjoint union and cartesian product. The associated Grothendieck ring $\mathrm{B}_{\mathbb{Z}}(H)$ is called the Burnside ring of H. The Burnside algebra of H over $\mathbb{K}$ is the $\mathbb{K}$-algebra $\mathrm{B}_{\mathbb{K}}^G(H) = \mathbb{K} \otimes_{\mathbb{Z}} \mathrm{B}_{\mathbb{Z}}(H)$. Therefore, letting V runs over representatives of the conjugacy classes of subgroups of H, then $[H/V]$ comprise (without repetition) a $\mathbb{K}$-basis of $\mathrm{B}_{\mathbb{K}}^G(H)$, where the notation $[H/V]$ denotes the isomorphism class of transitive H-sets whose stabilizers are H-conjugates of V. The maps on $\mathrm{B}_{\mathbb{K}}^G$ are given as follows:

$$t_H^K\bigl([H/V]\bigr) = [K/V], \qquad r_H^K\bigl([K/W]\bigr) = \sum_{HgW \subseteq K} \bigl[H/H \cap {}^gW\bigr], \qquad c_H^g\bigl([H/U]\bigr) = \bigl[{}^gH/{}^gU\bigr].$$

For any prime number p and any natural number n we write n_p to denote the p-part of n.

Theorem 7.1. *Let $M = \mathrm{B}_{\mathbb{K}}^G$, and let H and K be subgroups of G. For any subgroup L of G we put $M_L = (M :_{e_L} b_L(M))$ where $e_L = t_L^L$. Then:*

(1) *Any maximal $\mu_{\mathbb{K}}(G)$-submodule of M is of the form M_L for some subgroup L of G.*
(2) *If M_H is a maximal $\mu_{\mathbb{K}}(G)$-submodule of M then $M/M_H \cong S^G_{H,\mathbb{K}}$.*
(3) *$M_H = M_K$ if and only if $H =_G K$.*
(4) *If $\mathbb{K}$ is of characteristic 0 then M_H is a maximal $\mu_{\mathbb{K}}(G)$-submodule of M.*
(5) *Let $\mathbb{K}$ be of characteristic $p > 0$. Then:*
 (i) *M_H is a maximal $\mu_{\mathbb{K}}(G)$-submodule of M if and only if $|N_G(H):H|_p = 1$.*
 (ii)
$$M/\operatorname{Jac}(M) \cong \bigoplus_{L \leqslant_G G:\ |N_G(L):L|_p=1} S^G_{L,\mathbb{K}}.$$
 (iii)
$$\operatorname{Jac}(M)(K) = \bigcap_{X \leqslant K:\ |N_G(X):X|_p=1} \big\{x \in M(K):\ r^K_X(x) \in b_X(M)\big\}.$$

Proof. (1) and (2) It follows from the relations $[L/V] = t^L_V([V/V])$ and $c^g_H([L/V]) = [{}^gL/{}^gV]$ that $\overline{M}(L) \cong \mathbb{K}$, as $\mathbb{K}\overline{N}_G(L)$-modules, for any subgroup L of G. The result follows by 4.4.

(3) It follows by part (2) of 4.5.

(4) In this case the Mackey algebra is semisimple by [10], and so the result follows by part (2) of 4.6.

(5) Using the first three parts we see that M_H is maximal if and only if $S^G_{H,\mathbb{K}}$ appears in the head of M. For any $X > H$, as $r^X_H([X/X]) = [H/H]$ we see that $r^X_H(M(X)) + b_H(M) = M(H)$. Thus, if p divides $|N_G(H):H|$ then 5.20 implies that $S^G_{H,\mathbb{K}}$ does not appear in the head of M. On the other hand, if p does not divide $|N_G(H):H|$ then part (4) of 5.19 implies that the multiplicity of $S^G_{H,\mathbb{K}}$ in the head of M is 1. These finish the proofs of parts (i) and (ii).

$\operatorname{Jac}(M)$ is the intersection of subfunctors M_X where X ranges over all subgroups X of G such that p does not divide $|N_G(X):X|$. Therefore, $x \in \operatorname{Jac}(M)(X)$ if and only $x \in M_X$ for any such subgroup X. The desired result follows by part (2) of 4.5. □

If we assume that $\mathbb{K}$ is algebraically closed then part (5)(ii) of 7.1 follows also by [11, (8.9) Corollary] which express $\mathrm{B}^G_{\mathbb{K}}$ as a direct sum of principal indecomposable $\mu_{\mathbb{K}}(G)$-modules.

Proposition 7.2. *Let $\mathbb{K}$ be of characteristic $p > 0$ and $M = \mathrm{B}^G_{\mathbb{K}}$. For any natural number k we put $J_k = \operatorname{Jac}^k(M)$. Let n be a natural number. Then:*

(1) *$J_n(H) = M(H)$ for any p-subgroup H of G with $|G:H|_p \geqslant p^n$.*
(2) *$J_n(H) = b_H(M)$ for any p-subgroup H of G with $|G:H|_p = p^{n-1}$, where $n \geqslant 1$.*
(3) *$J_n(H) = b_H(M)$ for any p-subgroup H of G with $|G:H|_p = p^{n-2}$, where $n \geqslant 2$.*
(4) *$J_{n+1}(H) = J_n(H)$ for any p-subgroup H of G with $|G:H|_p = p^{n-1}$, where $n \geqslant 1$.*

Proof. (1) Part (5)(iii) of 7.1 shows that the result is true for $n = 1$. Assume that the result is true for n. Take any p-subgroup K of G with $|G:K|_p \geqslant p^{n+1}$. Our aim is to show that $J_{n+1}(K) = M(K)$. As $J_n(K) = M(K)$, we see that $J_{n+1}(K) = M(K)$ if and only if evaluation of any simple summand of J_n/J_{n+1} at K is 0. Let $S^G_{L,U}$ be a simple summand of J_n/J_{n+1}. If $S^G_{L,U}(K) \neq 0$ then $L \leqslant_G K$ so that L is a p-subgroup of G with $|G:L|_p \geqslant p^{n+1}$. We will finish the proof by showing that there is no simple functor in the head of J_n that has X as a minimal subgroup where X is a p-subgroup of G with $|G:X|_p \geqslant p^{n+1}$. Let X be such a subgroup. It is clear that $J_n(X) = M(X)$ and $b_X(J_n) = b_X(M)$, and that $J_n(Y) = M(Y)$ for any $Y > X$ with $|Y:X| = p$. As $\overline{J_n}(X) \cong \mathbb{K}$, we see by using 4.4 that if $S^G_{X,V}$ appears in the head of J_n for some simple $\mathbb{K}\overline{N}_G(X)$-module V, then $V = \mathbb{K}$. Thus, it follows by 5.20

that the multiplicities of $S^G_{X,\mathbb{K}}$ in the heads of J_n and M are equal. But p divides $|N_G(X):X|$, and so by 7.1 we see that $S^G_{X,\mathbb{K}}$ does not appear in the head of M.

(2) The result is true for $n=1$ by part (5)(iii) of 7.1. Assume that the result is true for n. Take any p-subgroup K of G with $|G:K|_p = p^n$. We want to show that $J_{n+1}(K) = b_K(M)$. Using part (1) we see that $J_n(K) = M(K)$ and $b_K(J_n) = b_K(M)$. Let $X > K$ with $|X:K| = p$. Then $J_n(X) = b_X(M)$ by the assumption of the result for n. We calculate easily that $r^X_K(J_n(X)) = r^X_K(b_X(M)) \subseteq b_K(M)$, and so $r^X_K(\overline{J_n}(X)) = 0$. Thus, 5.20 implies that $S^G_{K,\mathbb{K}}$ appears in the head of J_n. As $\overline{J_n}(K) = \overline{M}(K) \cong \mathbb{K}$ and as $b_K(J_n) = b_K(M)$, we deduce by 4.4 that J_n has a unique maximal subfunctor I whose simple quotient has K as a minimal subgroup, and that I satisfies $I(K) = b_K(M)$.

For any p-subgroup Y of G with $|G:Y|_p \geqslant p^{n+1}$ it follows by part (1) that $\overline{J_n}(Y) \cong \mathbb{K}$ so that any simple functor having Y as a minimal subgroup and appearing in the head of J_n must be of the form $S^G_{Y,\mathbb{K}}$. Now 5.20 implies that the multiplicity of $S^G_{Y,\mathbb{K}}$ in the heads of J_n and M are equal. Thus, 7.1 gives that J_n has no simple functor in its heads with a minimal subgroup Y satisfying $|G:Y|_p \geqslant p^{n+1}$. Consequently, if J is a maximal subfunctor of J_n whose simple quotient J_n/J is nonzero at K, then J must be equal to I. Hence, $J_{n+1}(K) = I(K) = b_K(M)$ because J_{n+1} is the intersection of maximal subfunctors of J_n.

(3) We first show that the result is true for $n=2$: Let H be a p-subgroup of G with $|G:H|_p = 1$. By part (2) we obtain that $J_1(H) = b_H(M)$. For any p-subgroup X of G such that $|G:X|_p \geqslant p$, part (1) gives that $J_1(X) = M(X)$, in particular, $\overline{J_1}(H) = 0$ and $\overline{J_1}(X) \cong \mathbb{K}$. Thus 4.4 implies that if a simple functor whose minimal subgroup is a p-group appears in the head of J_n then it must be of the form $S^G_{X,\mathbb{K}}$ where X is a p-subgroup with $|G:X|_p \geqslant p$. Using 5.20 we see easily that the simple functors in the head of J_1 whose minimal subgroups are p-groups are precisely of the form $S^G_{K,\mathbb{K}}$ where K ranges over all subgroups of G with $|G:K|_p = p$. Now 6.18 implies that the evaluation of J_1/J_2 at H is 0. Hence, $J_2(H) = J_1(H) = b_H(M)$.

Assume that the result is true for n. Take a subgroup K of G with $|G:K|_p = p^{n-1}$. We want to justify that $J_{n+1}(K) = b_K(M)$. Then $J_n(K) = b_K(M)$ by part (2), and $b_Z(J_n) = b_Z(M)$ for any p-subgroup Z with $|G:Z|_p \geqslant p^n$ by part (1). As in the first paragraph proving the result for $n=2$, we may see that the simple functors in the head of J_n whose minimal subgroups are p-groups in some conjugate of K are of the form $S^G_{A,\mathbb{K}}$ where A are some subgroups of G with $|G:A|_p = p^n$. Thus, applying 6.18 again we see that the value of J_n/J_{n+1} at K is zero. Therefore, $J_{n+1}(K) = J_n(K) = b_K(M)$ where the last equality follows from part (2).

(4) It follows by parts (2) and (3) that they both equal to $b_H(M)$. $\square$

Theorem 7.3. *Let $\mathbb{K}$ be of characteristic $p > 0$ and $M = \mathrm{B}^G_{\mathbb{K}}$. Let H be a p-subgroup of G, and V be a simple $\mathbb{K}\overline{N}_G(H)$-module, and let n be a natural number with $p^n \leqslant |G|_p$. For any natural number k we put $J_k = \mathrm{Jac}^k(M)$. Then*:

1. *If $S^G_{H,V}$ appears in J_n/J_{n+1} then $|G:H|_p \leqslant p^n$ and $|G:H|_p \neq p^{n-1}$.*
2. *If $|G:H|_p = p^n$ and $S^G_{H,V}$ appears in J_n/J_{n+1} then $V = \mathbb{K}$.*
3. *If $|G:H|_p = p^n$ then the multiplicity of $S^G_{H,\mathbb{K}}$ in J_n/J_{n+1} is 1.*
4. *The multiplicity of $S^G_{1,\mathbb{K}}$ in M is 1, and it appears in J_m/J_{m+1} where $p^m = |G|_p$. In particular, the Loewy length of M is greater than or equal to $m+1$.*

Proof. (1) If $|G:H|_p \geqslant p^{n+1}$ or $|G:H|_p = p^{n-1}$ then by 7.2 we obtain that $J_n(H) = J_{n+1}(H)$. Thus the result follows.

(2) It follows by 7.2 that $\overline{J_n}(H) \cong \mathbb{K}$. The conclusion $V = \mathbb{K}$ follows from 4.4.

(3) Let $|G:H|_p = p^n$ and let $X > H$ with $|X:H| = p$. Then 7.2 gives that $J_n(X) = b_X(M)$, $J_n(H) = M(H)$, and $b_H(J_n) = b_H(M)$. It is easy to see that $r^X_H(J_n(X)) = r^X_H(b_X(M)) \subseteq b_H(M)$. Now, 5.20 shows that the multiplicity of $S^G_{H,\mathbb{K}}$ in J_n/J_{n+1} is 1.

(4) As $M(1) \cong \mathbb{K}$ as $\mathbb{K}G$-module, it is clear that the multiplicity of $S^G_{1,\mathbb{K}}$ in M is 1 (see also 6.8). Moreover, we see by part (3) that $S^G_{1,\mathbb{K}}$ appears in J_m/J_{m+1} where $p^m = |G|_p$. $\square$

The following result may be obtained by using the previous two results and their proofs.

Corollary 7.4. *Let $\mathbb{K}$ be of characteristic $p > 0$ and G be a p-group with $|G| \geqslant p^3$. For any natural number k we put $J_k = \mathrm{Jac}^k(M)$ where $M = \mathrm{B}^G_{\mathbb{K}}$. Then:*

$$J_2/J_3 \cong \left(\bigoplus_{H \leqslant_G G:\ |G:H|=p^2} S^G_{H,\mathbb{K}} \right) \bigoplus \lambda S^G_{G,\mathbb{K}},$$

$$J_3/J_4 \cong \left(\bigoplus_{H \leqslant_G G:\ |G:H|=p^3} S^G_{H,\mathbb{K}} \right) \oplus \left(\bigoplus_{H \leqslant_G G:\ |G:H|=p} \lambda_H S^G_{H,\mathbb{K}} \right),$$

where λ is the number of elements of the set $\{V \leqslant_G G\colon |G:V| = p\}$, and λ_H is the number of $\overline{N}_G(H)$-orbits on the set $\{V \leqslant_H H\colon |H:V| = p\}$ on which $\overline{N}_G(H)$ acts by conjugation.

In the case of the previous result, one sees that $J_{k+1}(X) = b_X(J_k)$ for any $k \in \{0, 1, 2\}$ and any $X \leqslant G$ with $|G:X| \leqslant p^k$. However, this may not be true for $k \geqslant 3$ unless G is abelian.

Lemma 7.5. *Let $\mathbb{K}$ be of characteristic $p > 0$ and G be an abelian p-group with $|G| \geqslant p^3$. For any natural number k we put $J_k = \mathrm{Jac}^k(M)$ where $M = \mathrm{B}^G_{\mathbb{K}}$. Then, $J_{n+1}(H) = b_H(J_n)$ for any n and any $H \leqslant G$ with $|G:H| \leqslant p^n \leqslant |G|$.*

Proof. Let X be a subgroup of G. As G is abelian, $\overline{N}_G(X)$ acts on $M(X)$ trivially so that $M(X)$, and hence each nonzero quotient of each $J_k(X)$, is a semisimple $\mathbb{K}\overline{N}_G(X)$-module. Then, 6.20 shows that $J_{k+1}(X) = b^0_X(J_k)$ for any k and any X.

We will prove the result by induction on n. It may be seen easily by using 7.2 that the result is true for $n = 0, 1$. Assume that the result is true for n. Take any subgroup K of G with $|G:K| \leqslant p^{n+1}$. We want to obtain that $J_{n+2}(K) = b_K(J_{n+1})$.

By the above, $J_{n+2}(K) = b^0_K(J_{n+1})$. Let $Y > K$ with $|Y:K| = p$. Then, $|G:Y| \leqslant p^n$ implying by the assumption of the result for n that $J_{n+1}(Y) = b_Y(J_n)$. Using the Mackey axiom we see that

$$r^Y_K\big(J_{n+1}(Y)\big) = \sum_{Z<Y:\ Y=KZ} t^K_{K\cap Z} r^Z_{K\cap Z}\big(J_n(Z)\big).$$

From the condition $Y = KZ$ it follows that $K \cap Z < Z$ and $K \cap Z < K$. As J_n/J_{n+1} is semisimple, 6.18 implies that the element $r^Z_{K\cap Z}$ of $\mu_{\mathbb{K}}(G)$ annihilates J_n/J_{n+1}. This gives that $r^Z_{K\cap Z}(J_n(Z)) \subseteq J_{n+1}(K \cap Z)$. Therefore,

$$r^Y_K\big(J_{n+1}(Y)\big) \subseteq \sum_{Z<Y:\ Y=KZ} t^K_{K\cap Z}\big(J_{n+1}(K\cap Z)\big) \subseteq b_K(J_{n+1}).$$

Consequently, $b^0_K(J_{n+1}) = b_K(J_{n+1})$ proving that $J_{n+2}(K) = b_K(J_{n+1})$. $\square$

For any rational number r we denote by $\lfloor r \rfloor$ the largest integer which is less than or equal to r.

Theorem 7.6. *Let $\mathbb{K}$ be of characteristic $p > 0$ and G be an abelian p-group. Let H be a subgroup of G with $|G:H| = p^m$ and n be a natural number with $m \leqslant n - 1$ and $p^n \leqslant |G|$. For any natural number k we put $J_k = \mathrm{Jac}^k(M)$ where $M = \mathrm{B}^G_{\mathbb{K}}$. Then:*

(1)

$$J_n(H)=\bigoplus_{V\leqslant H:\ |H:V|\geqslant p^{s+1}}\mathbb{K}[H/V],$$

where $s=\lfloor (n-m-1)/2\rfloor$.

(2) $S^G_{H,\mathbb{K}}$ *does not appear in* J_n/J_{n+1} *if and only if* $n-m$ *is an odd number.*

(3) *Suppose that* $n-m$ *is an even number. Then, the multiplicity of* $S^G_{H,\mathbb{K}}$ *in* J_n/J_{n+1} *is equal to the number of elements of the set* $\{V\leqslant H:\ |H:V|=p^{(n-m)/2}\}$.

Proof. (1) For any nonnegative integer i, we see that $i\leqslant s$ if and only if $m+i\leqslant n-i-1$. Thus, if $i\leqslant s$ then we get by 7.5 that $J_{n-i}(X)=b_X(J_{n-i-1})$ for any $X\leqslant G$ with $|G:X|\leqslant p^{n-i-1}$. Moreover, by the transitivity of trace maps on a Mackey functor (i.e, $t^A_Bt^B_C=t^A_C$ for $C\leqslant B\leqslant A$) we see that $b_K(M)$ is the sum of $\mathbb{K}$-subspaces of $M(K)$ of the form $t^K_L(M(L))$ where L ranges over all subgroups of K satisfying $|L:K|=p$.

The result will follow by repeated applications of 7.5. To illustrate it, assuming $s\geqslant 2$, we see that

$$\begin{aligned}
J_n(H)&=b_H(J_{n-1})\\
&=\sum_{X_1\leqslant H:\ |H:X_1|=p}t^H_{X_1}\big(J_{n-1}(X_1)\big)\\
&=\sum_{X_1\leqslant H:\ |H:X_1|=p}t^H_{X_1}\big(b_{X_1}(J_{n-2})\big)\\
&=\sum_{X_1\leqslant H:\ |H:X_1|=p}t^H_{X_1}\sum_{X_2\leqslant X_1:\ |X_1:X_2|=p}t^{X_1}_{X_2}\big(J_{n-2}(X_2)\big)\\
&=\sum_{X_2\leqslant H:\ |H:X_2|=p^2}t^H_{X_2}\big(J_{n-2}(X_2)\big).
\end{aligned}$$

By the explanation given in the first paragraph of the proof we can apply 7.5 to $J_n(H)$ as above s-times to obtain

$$J_n(H)=\sum_{Y\leqslant H:\ |H:Y|=p^s}t^H_Y\big(J_{n-s}(Y)\big).$$

It is clear that $n-m-2\leqslant 2s\leqslant n-m-1$ so that $(n-s)-2\leqslant m+s\leqslant (n-s)-1$. As $|G:Y|=p^{m+s}$ we must have by 7.2 that $J_{n-s}(Y)=b_Y(M)$. Hence, the result follows.

(2) It is a consequence of 6.18 that $S^G_{H,\mathbb{K}}$ does not appear in J_n/J_{n+1} if and only if $J_n(H)=J_{n+1}(H)$, which is, by part (1), equivalent to the requirement that $\lfloor (n-m-1)/2\rfloor=\lfloor (n-m)/2\rfloor$. The result is clear now.

(3) Suppose that $n-m$ is even. Then, 6.18 implies that the multiplicity of $S^G_{H,\mathbb{K}}$ in J_n/J_{n+1} is equal to the dimension of $J_n(H)/J_{n+1}(H)$. The result now follows by part (1). □

By part (1) of 7.6 we know the evaluations $J_n(H)$ where G is an abelian p-group, n is a natural number with $p^n\leqslant |G|$, and H is a subgroup of G with $|G:H|\leqslant p^{n-1}$. For a subgroup H of G with $|G:H|\geqslant p^n$ we already knew by part (1) of 7.2 that $J_n(H)=M(H)$. Moreover, if $|G:H|\geqslant p^n$ then the integer s in part (1) of 7.6 is a negative integer so that every subgroup V of H satisfies $|H:V|\geqslant p^{s+1}$. The conclusion is that we can drop the condition $m\leqslant n-1$ from the hypothesis of 7.6.

The following is an immediate consequence of 7.3 and 7.6.

Corollary 7.7. *Let $\mathbb{K}$ be of characteristic $p > 0$ and G be an abelian p-group. For any natural number k we put $J_k = \mathrm{Jac}^k(M)$ where $M = \mathrm{B}^G_{\mathbb{K}}$. Then, for any natural number n with $p^n \leqslant |G|$, we have*

$$J_n/J_{n+1} \cong \bigoplus_{l=0}^{\lfloor n/2\rfloor}\Bigl(\bigoplus_{H\leqslant G:\ |G:H|=p^{n-2l}} \lambda^l_H S^G_{H,\mathbb{K}}\Bigr)$$

where λ^l_H is the number of elements of the set $\{V \leqslant H:\ |H : V| = p^l\}$.

Let G be an abelian p-group with $|G| = p^n$. To study the radical factors J_{n+r}/J_{n+r+1} of $\mathrm{B}^G_{\mathbb{K}}$, where $r \geqslant 1$, we first extend 7.5 to other cases.

Lemma 7.8. *Let $\mathbb{K}$ be of characteristic $p > 0$ and G be an abelian p-group with $|G| = p^n$. Let $r \geqslant 1$ be a natural number. For any natural number k we put $J_k = \mathrm{Jac}^k(M)$ where $M = \mathrm{B}^G_{\mathbb{K}}$. Then, $J_{n+r}(H) = b_H(J_{n+r-1})$ for any subgroup H of G.*

Proof. The result is true for $r = 1$ by 7.5. Assume that the result is true for r. As each $M(H)$ is a semisimple $\mathbb{K}\overline{N}_G(H)$-module, it follows by 6.20 that $J_{n+r+1}(H) = b^0_H(J_{n+r})$. It can be seen by arguing as in the proof of 7.5 that $b^0_H(J_{n+r}) = b_H(J_{n+r})$. □

The radical factors of $\mathrm{B}^G_{\mathbb{K}}$ not covered in 7.6 is the content of the next result.

Theorem 7.9. *Let $\mathbb{K}$ be of characteristic $p > 0$ and G be an abelian p-group with $|G| = p^n$. Let H be a subgroup of G with $|G : H| = p^m$. For any natural number k we put $J_k = \mathrm{Jac}^k(M)$ where $M = \mathrm{B}^G_{\mathbb{K}}$. Then:*

(1)

$$J_k(H) = \bigoplus_{V\leqslant H:\ |H:V|\geqslant p^{s+1}} \mathbb{K}[H/V],$$

where $s = \lfloor (k - m - 1)/2\rfloor$.

(2) *Assume that $k \geqslant n + 1$. Then, $S^G_{H,\mathbb{K}}$ appears in J_k/J_{k+1} if and only if $k - m$ is an even number satisfying $(k - m)/2 \leqslant (n - m)$. Moreover, in this case, the multiplicity of $S^G_{H,\mathbb{K}}$ in J_k/J_{k+1} is equal to the number of elements of the set $\{V \leqslant H:\ |H : V| = p^{(k-m)/2}\}$.*

Proof. (1) We may assume that $k = n + r$ where $r \geqslant 1$ is a natural number, because the result is true for $k \leqslant n$ by the virtue of part (1) of 7.6. It follows by repeated applications of 7.8 that

$$J_{n+r}(H) = \sum_{X\leqslant H:\ |H:X|=p^r} t^H_X\bigl(J_n(X)\bigr).$$

Then, part (1) of 7.6 implies that

$$J_{n+r}(H) = \bigoplus_{V\leqslant H:\ |H:V|=p^{s'+r+1}} \mathbb{K}[H/V]$$

where $s' = \lfloor (n - m - r - 1)/2\rfloor$. The result follows because $s' + r = \lfloor (n + r - m - 1)/2\rfloor$.

(2) It follows by 6.18 that $S^G_{H,\mathbb{K}}$ appears in J_k/J_{k+1} if and only if $J_k(H) \neq J_{k+1}(H)$. Note also that if $J_k(H) \neq J_{k+1}(H)$ then $J_k(H) \neq 0$ so that $|H| \geqslant p^{s+1}$ by part (1). Therefore, part (1) gives

the equivalence of $J_k(H) \neq J_{k+1}(H)$ to the conditions $\lfloor (k-m-1)/2 \rfloor \neq \lfloor (k-m)/2 \rfloor$ and $(n-m) \geqslant \lfloor (k-m-1)/2 \rfloor$. The result now follows easily. $\square$

The following obvious consequence of 7.9 deals with the cases not contained in 7.7.

Corollary 7.10. *Let $\mathbb{K}$ be of characteristic $p > 0$ and G be an abelian p-group with $|G| = p^n$. For any natural number k we put $J_k = \mathrm{Jac}^k(M)$ where $M = \mathrm{B}_{\mathbb{K}}^G$. Then, for any $k \geqslant n+1$,*

$$J_k/J_{k+1} \cong \bigoplus_{l=k-n}^{\lfloor k/2 \rfloor} \Bigl(\bigoplus_{H \leqslant G:\ |G:H| = p^{k-2l}} \lambda_H^l S_{H,\mathbb{K}}^G \Bigr)$$

where λ_H^l is the number of elements of the set $\{V \leqslant H:\ |H:V| = p^l\}$. In particular, the Loewy length of M is $2n+1$.

Let $\mathbb{K}$ be of characteristic $p > 0$ and G be a p-group with $|G| = p^n$. Then, $t_1^G r_1^G(\mathrm{B}_{\mathbb{K}}^G(G)) \neq 0$ because $t_1^G r_1^G([G/G]) = [G/1]$. The proof of 6.22 shows that the Loewy length of $\mathrm{B}_{\mathbb{K}}^G$ is greater than or equal to $2n+1$.

Let $\mathbb{K}$ be of characteristic $p > 0$ and G be an abelian p-group with $|G| = p^n$. One may see (for instance) by using 7.7 and 7.10 that, up to multiplicities of simple functors in the radical layers, the shape of the diagram showing the factors of the radical series of $\mathrm{B}_{\mathbb{K}}^G$ is symmetric. More precisely, for any natural number r with $1 \leqslant r \leqslant n$, a simple functor S appears in J_{n-r}/J_{n-r+1} if and only if S appears in J_{n+r}/J_{n+r+1}.

We next want to study the socle series of $\mathrm{B}_{\mathbb{K}}^G$ where $\mathbb{K}$ is of characteristic $p > 0$ and G is a (abelian) p-group. This turns out to be harder than the study of the radical series we presented in this section because determination of restriction kernels of $\mathrm{B}_{\mathbb{K}}^G$ is much harder than determination of Brauer quotients of $\mathrm{B}_{\mathbb{K}}^G$, all of which were isomorphic to trivial modules.

For any finite group H we use the notation $\Phi(H)$ to denote the Frattini subgroup of H which is the intersection of all maximal subgroups of H. It is the set of all nongenerators of H so that $\Phi(H)X \neq H$ for any proper subgroup X of H.

Lemma 7.11. *Let $\mathbb{K}$ be of characteristic $p > 0$ and G be a p-group. Put $M = \mathrm{B}_{\mathbb{K}}^G$. For any subgroups K and L of G, we have*:

(1) *If $K \leqslant L$ with $|L:K| = p$ then*

$$\mathrm{Ker}\bigl(r_K^L : M(L) \to M(K)\bigr) \subseteq \Bigl(\bigoplus_{V \leqslant_L L:\ N_L(V) \nleqslant K} \mathbb{K}[L/V] \Bigr).$$

(2)

$$\Bigl(\bigoplus_{V \trianglelefteq L:\ V \leqslant \Phi(L)} \mathbb{K}[L/V] \Bigr) \subseteq \underline{M}(L) \subseteq \Bigl(\bigoplus_{V \trianglelefteq L} \mathbb{K}[L/V] \Bigr).$$

(3)

$$\bigl(k_L^0(M)\bigr)^{\overline{N}_G(L)} \subseteq \Bigl(\bigoplus_{V \leqslant_L L:\ N_G(V) = L} \mathbb{K}[L/V] \Bigr).$$

Proof. (1) Let $K \trianglelefteq L$ with $|L:K| = p$. Any subgroup V of L satisfies exactly one of the three conditions: $N_L(V) \leqslant K$; $V \leqslant K \ngeqslant N_L(V)$; $V \nleqslant K$. As these conditions closed under taking L-conjugates of V, we can write the set of L-conjugacy classes of subgroups of L as a disjoint union of the three sets: $\mathcal{B}_1 = \{V \leqslant_L L\colon\ N_L(V) \leqslant K\}$, $\mathcal{B}_2 = \{V \leqslant_L L\colon\ V \leqslant K \ngeqslant N_L(V)\}$, and $\mathcal{B}_3 = \{V \leqslant_L L\colon\ V \nleqslant K\}$. Thus, letting

$$B_i = \bigoplus_{V \in \mathcal{B}_i} \mathbb{K}[L/V],$$

we may write $M(L) = B_1 \oplus B_2 \oplus B_3$ as $\mathbb{K}$-spaces. Using the definitions of restriction maps on M it is easy to verify the three properties:

$$r^L_K : B_1 \to M(K) \text{ is injective}; \qquad r^L_K(B_2) = 0; \qquad r^L_K(B_1) \cap r^L_K(B_3) = 0.$$

Now, let $x \in M(L)$ and write $x = x_1 + x_2 + x_3$ where $x_i \in B_i$ for each i. If $r^L_K(x) = 0$ then it follows by the above properties that $x_1 = 0$. This completes the proof.

(2) Let $x \in \underline{M}(L)$. Assume that there is a nonnormal subgroup V of L such that $[L/V]$ appears in x with nonzero coefficient. We can choose a maximal subgroup K of L containing $N_L(V)$. Then $|L:K| = p$ and $x \in \operatorname{Ker} r^L_K$. But this is impossible by part (1). The other inclusion is obvious.

(3) Let $x \in (k^0_L(M))^{\overline{N}_G(L)}$. Take a subgroup $\overline{X} = X/L$ of $\overline{N}_G(L)$ of order p. Then,

$$x \in \operatorname{Ker}\bigl(t^X_L : \underline{M}(L)^{\overline{X}} \to M(X)\bigr)$$

(see 6.21). It follows by part (2) that $x \in \operatorname{Ker}(t^X_L : U^{\overline{X}} \to M(X))$ where

$$U = \bigoplus_{V \trianglelefteq L} \mathbb{K}[L/V].$$

The $\mathbb{K}\overline{X}$-module U is a permutation module with a permutation basis $S = \{[L/V]\colon\ V \trianglelefteq L\}$. The $\overline{X}$-orbit sums of S form a $\mathbb{K}$-basis of $U^{\overline{X}}$. As the order of $\overline{X}$ is p, the sizes of $\overline{X}$-orbits of S are 1 or p. It is obvious that the image under t^X_L of any orbit sum of size p is 0. Furthermore, if V and W are normal subgroups of L such that $N_X(V) = X = N_X(W)$ (equivalently, the sizes of orbits containing each are both equal to 1) then $t^X_L([L/V]) = [X/V]$ and $t^X_L([L/W]) = [X/W]$ are distinct basis elements of $M(X)$. If we write x as a linear combination of $\overline{X}$-orbit sums of S then we see that the coefficient of any orbit sum of size 1 must be 0. Therefore, x can be written as a linear combination of elements of $M(L)$ of the form $[L/V]$ with $N_X(V) = L$.

To finish, if $[L/V]$ with $V \trianglelefteq L$ and with $N_G(V) \neq L$ appears in x, then we may choose a subgroup of Y/L of $N_G(V)/L$ of order p. Then $N_Y(V) = Y$, which is impossible, because what we have observed above implies that $N_Y(V) = L$. $\square$

Proposition 7.12. *Let $\mathbb{K}$ be of characteristic $p > 0$. Let G be a p-group and H be a subgroup of G. Put $M = \mathrm{B}^G_{\mathbb{K}}$. Then*:

(1) *If $S^G_{H,\mathbb{K}}$ appears in* $\operatorname{Soc}(M)$ *then $H = N_G(V)$ for some subgroup V of H.*
(2) *The multiplicity of $S^G_{G,\mathbb{K}}$ in* $\operatorname{Soc}(M)$ *is equal to* $\dim_{\mathbb{K}} \underline{M}(G)$*, which is nonzero.*
(3) $S^G_{N_G(\Phi(H)),\mathbb{K}}$ *appears in* $\operatorname{Soc}(M)$.
(4) *If G is abelian, then* $\operatorname{Soc}(M)(G) = \underline{M}(G)$ *and* $\operatorname{Soc}(M)(X) = 0$ *for any proper subgroup X of G.*

Proof. (1) Let S be a simple subfunctor of M such that S is isomorphic to $S^G_{H,\mathbb{K}}$. It follows by 6.21 that $S(H) \subseteq k^0_H(M)$. As $\overline{N}_G(H)$ acts on $S(H) \cong \mathbb{K}$ trivially, we must have that $S(H) \subseteq (k^0_H(M))^{\overline{N}_G(H)}$. In particular, $(k^0_H(M))^{\overline{N}_G(H)} \neq 0$. The result follows by part (3) of 7.11.

(2) It follows by part (2) of 7.11 and by 6.3.

(3) By part (2) we may assume that $N_G(\Phi(H)) \neq G$. For any subgroup V of G with $N_G(V) \neq G$ we put

$$x_V = \sum_{gN_G(V) \subseteq N_G(N_G(V))} \big[N_G(V)/{}^gV\big].$$

It is easy to see that an element $g \in N_G(N_G(V))$ satisfies $[N_G(V)/V] = [N_G(V)/{}^gV]$ if and only if $g \in N_G(V)$. This shows that $x_V \in M(N_G(V))^{N_G(N_G(V))}$. Take any $K \geqslant N_G(V)$ with $|K : N_G(V)| = p$. Then, $N_G(V) \trianglelefteqslant K \leqslant N_G(N_G(V))$ so that

$$x_V = \sum_{N_G(V)a \subseteq K} c^a_{N_G(V)} \bigg(\sum_{Kb \subseteq N_G(N_G(V))} \big[N_G(V)/{}^bV\big] \bigg),$$

implying that

$$t^K_{N_G(V)}(x_V) = \big|K : N_G(V)\big| \sum_{Kb \subseteq N_G(N_G(V))} \big[K/{}^bV\big] = 0.$$

Letting now $V = \Phi(H)$ and $L = N_G(V)$ we see by the above and by part (2) of 7.11 that $x_V = k^0_L(M)^{N_G(L)}$. Thus, $\mathbb{K}x_V$ is a $\mathbb{K}\overline{N}_G(L)$-submodule of $k^0_L(M)$ isomorphic to the trivial module $\mathbb{K}$, in particular it is simple. Hence, 6.21 implies that $S^G_{N_G(V),\mathbb{K}}$ appears in $\mathrm{Soc}(M)$.

(4) If $S^G_{H,\mathbb{K}}$ appears in $\mathrm{Soc}(M)$ then part (1) implies that $H = G$. The result follows by 6.21. $\square$

Part (1) of 7.12 is a special case of [7, Proposition 2.4], that can also be obtained by using it. Moreover, calculating the dimension of $\underline{M}(G)$, where $M = \mathrm{B}^G_{\mathbb{K}}$, is not easy even for small abelian p-groups. See [7, Section 3] where this dimension is calculated for some abelian p-groups.

As the Mackey algebra $\mu_{\mathbb{K}}(G)$ is not self-injective unless p^2 does not divide $|G|$ (see [11, (19.2) Theorem]), the socle of a principal indecomposable $\mu_{\mathbb{K}}(G)$-module $P^G_{H,V}$ may not be isomorphic to $S^G_{H,V}$. Thus, determination of the socle of a $\mu_{\mathbb{K}}(G)$-module of the form $P^G_{H,V}$ is not out of interest and studied in [7]. In particular, letting $\mathbb{K}$ be algebraically closed and G be a p-group, it is shown in [7, Proposition 2.4] by using a filtration of projective functors described in [12] that if $S^G_{K,\mathbb{K}}$ appears in $\mathrm{Soc}(P^G_{H,\mathbb{K}})$ then $K = N_H(L)$ for some $L \leqslant H$. In the general case, by the category equivalence described in [11, Section 10], finding $\mathrm{Soc}(P^G_{H,V})$ is equivalent to finding $\mathrm{Soc}(P^{\overline{N}_G(J)}_{H/J,V})$ where $J = O^p(H)$. Thus, to understand socles of principal indecomposable functors one has to find the socle of a $\mu_{\mathbb{K}}(G)$-module of the form $P^G_{H,V}$ where H is a p-group. Moreover, letting $\mathbb{K}$ be algebraically closed, we have by [11, (8.6) Theorem] that $P^G_{H,V}$ is a direct summand of $\uparrow^G_H \mathrm{B}^H_{\mathbb{K}}$. Therefore, studying the socle of the Burnside functor $\mathrm{B}^H_{\mathbb{K}}$ for a p-subgroup of H of G is important for the determination of the socle of $P^G_{H,V}$. Regarding this problem we only state the following.

Remark 7.13. Let $\mathbb{K}$ be an algebraically closed field of characteristic $p > 0$, and let K and H be subgroups of G. Suppose that W is a simple $\mathbb{K}\overline{N}_G(H)$-module and V is a simple $\mathbb{K}\overline{N}_G(K)$-module. Then:

(1) Assume that H is a p-subgroup of G. If a simple $\mu_{\mathbb{K}}(G)$-module S appears in the socle of $P^G_{H,W}$, then $S(N_H(L)) \neq 0$ for some $L \leqslant H$.

(2) Assume that H is a p-subgroup of G and $\dim_{\mathbb{K}} V = 1$. If $S^G_{K,V}$ appears in the socle of $P^G_{H,W}$, then $K =_G N_H(L)$ for some $L \leqslant H$.

(3) Assume that H is a normal p-subgroup of G. If $S^G_{K,V}$ appears in the socle of $P^G_{H,W}$, then $K = N_H(L)$ for some $L \leqslant H$.
(4) If $\overline{N}_G(H)$ is a p-group, then $S^G_{H,\mathbb{K}}$ appears in the socle of $P^G_{H,\mathbb{K}}$ with multiplicity equal to $\dim_{\mathbb{K}} \underline{T}(H)$, where $T = \mathrm{B}^H_{\mathbb{K}}$.
(5) Assume that H is a p-subgroup of G. Then, for any simple $\mathbb{K}\overline{N}_G(H)$-module U there is a simple $\mathbb{K}\overline{N}_G(H)$-module U' such that $S^G_{H,U}$ appears in the socle of $P^G_{H,U'}$.

Proof. (1) Let $B = \mu_{\mathbb{K}}(H)$ and $T = \mathrm{B}^H_{\mathbb{K}}$. Suppose that S appears in the socle of $P^G_{H,W}$. As $P^G_{H,W}$ is a direct summand of $\uparrow^G_H T$, it follows by the adjointness of the pair $(\downarrow^G_H, \uparrow^G_H)$ that $\mathrm{Hom}_B(\downarrow^G_H S, T) \neq 0$. Let $\mathcal{X} = \{N_H(X)\colon\ X \leqslant H\}$ and $e = e_{\mathcal{X}}$ be the idempotent of B defined as in 3.18 by $e_{\mathcal{X}} = \sum_{X \in \mathcal{X}} t^X_X$. Part (1) of 7.12 implies that T has no nonzero B-submodule annihilated by e. Then, by part (1) of 3.9, we see that $\mathrm{Hom}_{eBe}(eS, eT) \neq 0$. In particular $eS \neq 0$, implying the result.

(2) and (3) They follow by part (1) and by 6.18.

(4) Let $T = \mathrm{B}^H_{\mathbb{K}}$. Using 6.5 we see that the $\mathbb{K}\overline{N}_G(H)$-modules $\underline{(\uparrow^G_H T)}(H)$ and $n\mathbb{K}\overline{N}_G(H)$ are isomorphic where $n = \dim_{\mathbb{K}} \underline{T}(H)$.

It is clear that taking restriction kernels respects finite direct sums. Indeed, for any Mackey functor M for G, we have by part (2) of 5.3 that $\underline{M}(H) = (M :_f 0)(H)$ for some idempotent f. So, part (6) of 4.1 implies that taking restriction kernels respects finite direct sums. This fact is also immediate from the isomorphism $(L^-{}_{N_G(H)/H} \downarrow^G_{N_G(H)} M)(H/H) \cong \underline{M}(H)$ of $\mathbb{K}\overline{N}_G(H)$-modules, because the functors L^- and $\downarrow$ respect finite direct sums.

For a principal indecomposable $\mu_{\mathbb{K}}(G)$-module $P = P^G_{Y,U}$ it follows by 6.5 that if $\underline{P}(H) \neq 0$ then $H \leqslant_G Y$. Thus, using the formula [11, (8.6) Theorem] expressing $\uparrow^G_H T$ as a direct sum of principal indecomposable $\mu_{\mathbb{K}}(G)$-modules, we see that $\underline{(\uparrow^G_H T)}(H) \cong \underline{P^G_{H,\mathbb{K}}}(H)$. Hence, the multiplicity of $S^G_{H,\mathbb{K}}$ in the socle of $P^G_{H,\mathbb{K}}$ is equal by 6.3 to n.

(5) As $\downarrow^G_H S^G_{H,U} \cong (\dim_{\mathbb{K}} U) S^H_{H,\mathbb{K}}$ and as $S^H_{H,\mathbb{K}}$ appears in the socle of $\mathrm{B}^H_{\mathbb{K}}$ (by part (2) of 7.12), we see by using the adjointness of the pair $(\downarrow^G_H, \uparrow^G_H)$ that $S^G_{H,U}$ appears in the socle of $\uparrow^G_H \mathrm{B}^H_{\mathbb{K}}$. The result follows by using the formula [11, (8.6) Theorem] expressing $\uparrow^G_H \mathrm{B}^H_{\mathbb{K}}$ as a direct sum of principal indecomposable and by arguing as in part (4). $\square$

Lemma 7.14. *Let $\mathbb{K}$ be of characteristic $p > 0$ and G be an abelian p-group. Let H be a subgroup of G and n be a natural number. For any natural number k we put $S_k = \mathrm{Soc}^k(M)$ where $M = \mathrm{B}^G_{\mathbb{K}}$. Then:*

(1) $S_{n+1}(H)/S_n(H) = k^0_H(M/S_n)$.
(2) *If* $|G:H| \geqslant p^n$ *then* $S_n(H) = 0$.
(3) *If* $|G:H| = p^{n-1}$ *and* $n \geqslant 1$ *then* $S_n(H) = \underline{M}(H)$.
(4) *If* $|G:H| = p^{n-2}$ *and* $n \geqslant 2$ *then* $S_n(H) = \underline{M}(H)$.
(5) *If* $|G:H| \leqslant p^n \leqslant |G|$ *then* $S_{n+1}(H)/S_n(H) = \underline{(M/S_n)}(H)$.

Proof. (1) As G is abelian, $\overline{N}_G(H)$ acts on $M(H)$ trivially so that each submodule of each quotient of $M(H)$, in particular $k^0_H(M/S_n)$, is semisimple. The result follows by 6.21.

(2) The result is true for $n = 0, 1$ by 7.12. Assuming that the result is true for n, take a subgroup K of G such that $|G:K| \geqslant p^{n+1}$. We want to show that $S_{n+1}(K) = 0$. Let $H \geqslant K$ with $|H:K| = p$. Then, $S_n(H) = 0 = S_n(K)$ by the assumption of the result for n. As G is abelian, the map t^H_K on M, and hence on M/S_n, is injective. This means by 6.18 that $S_{K,\mathbb{K}}$ does not occur in S_{n+1}/S_n so that, by 6.18 again, $S_{n+1}(K) = S_n(K) = 0$.

(3) The result is true for $n = 1$ by 7.12. Assume that the result is true for n. Take a subgroup K of G with $|G:K| = p^n$. We want to show that $S_{n+1}(K) = \underline{M}(K)$. We will achieve this by first calculating $k^0_K(M/S_n)$ and then by using part (1).

Part (2) implies that

$$\underline{(M/S_n)}(K)=\{x\in M(K)\colon\ r^K_J(x)\in S_n(J),\ \forall J<K\}/S_n(K)=\underline{M}(K)/0.$$

Let $H\geqslant K$ with $|H:K|=p$. For any $x\in\underline{M}(K)$, we see by using the Mackey axiom that $r^H_Jt^H_K(x)=0$ for any $J<H$ so that $t^H_K(x)\in\underline{M}(H)=S_n(H)$. Hence,

$$k^0_K(M/S_n)=\underline{(M/S_n)}(K)=\underline{M}(K)/0.$$

As $S_n(K)=0$, the result follows by part (1).

(4) Using the first three parts we see that

$$k^0_G(M/S_1)=\underline{(M/S_1)}(G)=\underline{M}(G)/\underline{M}(G)=0$$

implying by 6.21 that $S^G_{G,\mathbb{K}}$ does not appear in S_2/S_1, and so $S_2(G)=S_1(G)=\underline{M}(G)$ by 6.18. Hence, the result is true for $n=2$. An easy induction argument on n finishes the proof.

(5) The result is true for $n=0$ because $S_1(G)=\underline{M}(G)$ by 7.12. Assume that the result is true for n. Take a subgroup K of G with $|G:K|\leqslant p^{n+1}$. Our aim is to obtain that $S_{n+2}(K)/S_{n+1}(K)=\underline{(M/S_{n+1})}(K)$.

We have by part (1) that $S_{n+2}(K)/S_{n+1}(K)=k^0_K(M/S_{n+1})$. Let $x\in M(K)$ be such that

$$x+S_{n+1}(K)\in\underline{(M/S_{n+1})}(K)=\{y\in M(K)\colon\ r^K_J(y)\in S_{n+1}(J),\ \forall J<K\}/S_{n+1}(K).$$

Then, $r^K_J(x)\in S_{n+1}(J)$ for any $J<K$. Take any $H\geqslant K$ with $|H:K|=p$. Then, for any $I<H$, it follows by the Mackey axiom that

$$r^H_It^H_K(x)=|H:IK|t^I_{I\cap K}r^K_{I\cap K}(x).$$

If $r^H_It^H_K(x)\neq 0$, then $H=IK$ implying that $I\cap K<I$ and $I\cap K<K$. It follows by 6.18 that the element $t^I_{I\cap K}$ of $\mu_{\mathbb{K}}(G)$ annihilates the semisimple functor S_{n+1}/S_n. This gives that $r^H_It^H_K(x)\in S_n(I)$, because $r^K_{I\cap K}(x)\in S_{n+1}(I\cap K)$. Therefore, $r^H_It^H_K(x)\in S_n(I)$ for every $I<H$, that means

$$t^H_K(x)+S_n(H)\in\{z\in M(H)\colon\ r^H_J(z)\in S_n(J),\ \forall J<H\}/S_n(H)=\underline{(M/S_n)}(H).$$

Now, the assumption of the result for n gives that $t^H_K(x)\in S_{n+1}(H)$. Consequently, any element $x+S_{n+1}(K)$ of $\underline{(M/S_{n+1})}(K)$ is mapped by t^H_K to the zero element of $M(H)/S_{n+1}(H)$. This yields that $k^0_K(M/S_{n+1})=\underline{(M/S_{n+1})}(K)$, as desired. □

Theorem 7.15. *Let $\mathbb{K}$ be of characteristic $p>0$ and G be an abelian p-group. Let H be a subgroup of G and n be a natural number with $p^n\leqslant|G|$. For any natural number k we put $S_k=\mathrm{Soc}^k(M)$ where $M=\mathrm{B}^G_{\mathbb{K}}$. Then:*

(1) *If $S^G_{H,\mathbb{K}}$ appears in S_{n+1}/S_n then $|G:H|\leqslant p^n$.*
(2) *If $|G:H|=p^{n-1}$ then $S^G_{H,\mathbb{K}}$ does not appear in S_{n+1}/S_n.*
(3) *If $|G:H|=p^n$ then the multiplicity of $S^G_{H,\mathbb{K}}$ in S_{n+1}/S_n is $\dim_{\mathbb{K}}\underline{M}(H)$.*
(4) *$S^G_{1,\mathbb{K}}$ appears in S_{m+1}/S_m where $p^m=|G|$.*

Proof. (1) and (2) They follow by parts (2)–(4) of 7.14.

(3) The multiplicity of $S^G_{H,\mathbb{K}}$ in S_{n+1}/S_n is equal by 6.18 to the dimension of $S_{n+1}(H)/S_n(H)$, that is isomorphic by 7.14 to $\underline{M}(H)$.

(4) Follows by part (3). □

Theorem 7.16. *Let $\mathbb{K}$ be of characteristic $p > 0$ and G be an abelian p-group. Let H be a subgroup of G with $|G:H| = p^m$ and n be a natural number with $m \leqslant n-1$ and $p^n \leqslant |G|$. For any natural number k we put $S_k = \mathrm{Soc}^k(M)$ where $M = \mathrm{B}^G_{\mathbb{K}}$. Then:*

(1)

$$S_n(H) = \bigcap_{X \leqslant H:\ |H:X| = p^{s+1}} \mathrm{Ker}\big(r^H_X : M(H) \to M(X)\big),$$

where $s = \lfloor (n-m-1)/2 \rfloor$.

(2) *If $n-m$ is an odd number then $S^G_{H,\mathbb{K}}$ does not appear in S_{n+1}/S_n.*

Proof. (1) For any subgroup K of G with $|G:K| \leqslant p^n$ it follows by part (5) of 7.14 that

$$\begin{aligned} S_{n+1}(K) &= \big\{x \in M(K):\ r^K_J(x) \in S_n(J),\ \forall J < K\big\} \\ &= \bigcap_{J \leqslant K:\ |K:J| = p} \big\{x \in M(K):\ r^K_J(x) \in S_n(J)\big\}. \end{aligned}$$

We will use this equality repeatedly to obtain the result. Arguing as in the proof of part (1) of 7.6, we apply the above equality s-times to $S_n(H)$ and obtain that

$$S_n(H) = \bigcap_{Y \leqslant H:\ |H:Y| = p^s} \big\{x \in M(H):\ r^H_Y(x) \in S_{n-s}(Y)\big\}.$$

As $|G:Y| = p^{m+s}$ and as $(n-s)-2 \leqslant m+s \leqslant (n-s)-1$, we see by parts (3) and (4) of 7.14 that $S_{n-s}(Y) = \underline{M}(Y)$. Thus, the result follows.

(2) It follows by the first part, because if $n-m$ is an odd number then $\lfloor (n-m-1)/2 \rfloor = \lfloor (n-m)/2 \rfloor$. □

The following is immediate from 7.16.

Corollary 7.17. *Let $\mathbb{K}$ be of characteristic $p > 0$ and G be an abelian p-group. For any natural number k we put $S_k = \mathrm{Soc}^k(M)$ where $M = \mathrm{B}^G_{\mathbb{K}}$. Then, for any natural number n with $p^n \leqslant |G|$ we have:*

$$S_{n+1}/S_n \cong \bigoplus_{l=0}^{\lfloor n/2 \rfloor} \Bigg(\bigoplus_{H \leqslant G:\ |G:H| = p^{n-2l}} \lambda^l_H S^G_{H,\mathbb{K}} \Bigg)$$

for some nonnegative integers λ^l_H.

Some of the numbers λ^l_H in 7.17 may be 0. For instance, letting G be the cyclic group of order p^4, one may calculate that $S_3/S_2 \cong 2S^G_{H,\mathbb{K}}$ where $|G:H| = p^2$, in particular, $S^G_{G,\mathbb{K}}$ does not appear in S_3/S_2.

Imitating the proofs of 7.8 and 7.9 one may obtain the following.

Theorem 7.18. *Let $\mathbb{K}$ be of characteristic $p > 0$ and G be an abelian p-group with $|G| = p^n$. Let H be a subgroup of G with $|G : H| = p^m$. For any natural number k we put $S_k = \mathrm{Soc}^k(M)$ where $M = \mathrm{B}^G_{\mathbb{K}}$. Then:*

(1)

$$S_k(H) = \bigoplus_{X \leqslant H:\ |H:X|=p^{s+1}} \mathrm{Ker}\big(r^H_X : M(H) \to M(X)\big),$$

where $s = \lfloor (k-m-1)/2 \rfloor$.

(2) *Assume that $k \geqslant n+1$. If $k-m$ is an odd number, then $S^G_{H,\mathbb{K}}$ does not appear in S_{k+1}/S_k.*

(3) *If $k \geqslant n+1$, then*

$$S_{k+1}/S_k \cong \bigoplus_{l=k-n}^{\lfloor k/2 \rfloor} \left(\bigoplus_{K \leqslant G:\ |G:K|=p^{k-2l}} \lambda^l_K S^G_{K,\mathbb{K}} \right)$$

for some nonnegative integers λ^l_K.

To give more applications of general results we obtained in previous sections we finish this section by studying the fixed point functor FP^G_V where V is a one dimensional $\mathbb{K}G$-module and $\mathbb{K}$ is of characteristic $p > 0$. As V is one dimensional, the $\mathbb{K}K$-module V is simple for any subgroup K of G, and if H is a p-subgroup of G then $V^H = V \neq 0$. Therefore, the image of the (relative) trace map t^K_H is 0 if $H < K$ are p-subgroups of G. Moreover, restrictions maps on a fixed point functor are all inclusions (so that injective), and in the case $\dim_{\mathbb{K}} V = 1$ we see if we assume $V^K \neq 0$ that the (relative) trace map t^K_H on FP^G_V is surjective if and only if p does not divide $|K : H|$.

Lemma 7.19. *Let $\mathbb{K}$ be of characteristic $p > 0$ and V be a one dimensional $\mathbb{K}G$-module. Let H be a subgroup of G and W be a simple $\mathbb{K}\overline{N}_G(H)$-module. Let J and S be $\mu_{\mathbb{K}}(G)$-submodules of M where $M = FP^G_V$. Then:*

(1) *$\overline{J}(H) \neq 0$ if and only if $J(H) = M(H)$ and H is a p-subgroup of G.*
(2) *$S^G_{H,W}$ appears in the head of J if and only if H is a maximal subgroup of G subject to the condition $\overline{J}(H) \neq 0$ and the $\mathbb{K}\overline{N}_G(H)$-module W is isomorphic to $V^H = V$.*
(3) *Any minimal subgroup of M/S is a p-subgroup of G.*
(4) *$\underline{(M/S)}(H) \neq 0$ if and only if H is a minimal subgroup of M/S.*

Proof. As $\dim_{\mathbb{K}} M(X) \leqslant 1$ for any subgroup X of G, we see that $J(X) \neq 0$ if and only if $J(X) = M(X) \neq 0$. We will use this trivial observation in the proof.

(1) This is trivial by the explanation given before 7.19.

(2) Suppose that $S^G_{H,W}$ appears in the head of J. By 4.4 the module W is isomorphic to a simple quotient module of the $\mathbb{K}\overline{N}_G(H)$-module $\overline{J}(H)$. As $\dim_{\mathbb{K}} M(Y) \leqslant 1$ for any $Y \leqslant G$, it is clear that if $\overline{J}(H) \neq 0$ then $\overline{J}(H) \cong M(H) = V^H = V$. In particular, $\dim_{\mathbb{K}} W = 1$ so that we may use 5.20. Assume that H is not maximal subject to the required condition. Then there is a $K > H$ satisfying $\overline{J}(K) \neq 0$. Using part (1) we can find a subgroup X with $H < X \leqslant K$ with $|X : H| = p$. Now $0 \neq r^K_H(J(K)) \subseteq r^X_H(J(X))$ implying that $r^X_H(J(X)) = J(H)$. But then 5.20 implies that $S^G_{H,W}$ does not appear in the head of J.

The converse implication follows by 6.3.

(3) Let X be a minimal subgroup of M/S. Then $M(X) \neq 0$, $S(X) = 0$ and $S(Y) = M(Y)$ for any $Y < X$. If X is not a p-group then $M(X) = t^X_Z(S(Z)) \subseteq S(X)$ where Z is a Sylow p-subgroup of X.

(4) Suppose that $\underline{(M/S)}(H) \neq 0$. Then $0 \neq r^H_X(M(H)) \subseteq S(X)$ for any $X < H$. Thus, $M(X) = S(X)$ for any $X < H$ implying that H is a minimal subgroup of M/S. $\square$

Theorem 7.20. *Let $\mathbb{K}$ be of characteristic $p > 0$ and V be a one dimensional $\mathbb{K}G$-module. For any natural number k we put $J_k = \mathrm{Jac}^k(M)$ and $S_k = \mathrm{Soc}^k(M)$ where $M = FP_V^G$. Let n be the natural number satisfying $p^n = |G|_p$. Then:*

(1)

$$J_k/J_{k+1} \cong \bigoplus_{H \leqslant_G G\colon |H| = p^{n-k}} S_{H,V}^G.$$

(2)

$$S_{k+1}/S_k \cong \bigoplus_{H \leqslant_G G\colon |H| = p^k} S_{H,V}^G.$$

(3) *The Loewy length of M is $n+1$.*
(4) *Let X be a p-subgroup of G. Then, $J_k(X) = 0$ if and only if $|X| \geqslant p^{n+1-k}$.*
(5) *Let X be a p-subgroup of G. Then, $S_k(X) = 0$ if and only if $|X| \geqslant p^k$.*
(6) *If G is a p-group then the socle and the radical series of M coincide.*

Proof. Firstly, as $\dim_{\mathbb{K}} M(X) \leqslant 1$ for any $X \leqslant G$, the multiplicity of any composition factor of M is 1.

(1) Parts (1) and (2) of 7.19 imply that $J_0/J_1 \cong S_{H,V}^G$ where $|H| = p^n$. Assume that the result is true for $k = 1, 2, \ldots, r$. Let K be a p-subgroup of G. Then, it follows by 6.18 that the evaluation of M/J_{r+1} at K is nonzero if and only if $|K| \geqslant p^{n-r}$. As $\dim_{\mathbb{K}} M(K) = 1$, we conclude that $J_{r+1}(K) = M(K)$ if and only if $|K| \leqslant p^{n-(r+1)}$. Therefore, parts (1) and (2) of 7.19 imply that the result is true for $k = r+1$.

(2) It may be justified as in part (1).

(3) It follows by part (1) or by part (2).

(4) As $\dim_{\mathbb{K}} M(X) = 1$, we see that $J_k(X) = 0$ if and only if the evaluation of M/J_k at X is nonzero. This is equivalent to the requirement that the evaluation of J_{k-m-1}/J_{k-m} at X is nonzero for some $m \geqslant 0$. Using part (1) and 6.18 we now conclude that $J_k(X) = 0$ if and only if $|X| = p^{n-(k-m-1)} \geqslant p^{n+1-k}$.

(5) It may be justified as in part (4).

(6) It follows by parts (4) and (5). $\square$

Acknowledgments

I would like to thank Peter Webb for his comments on the paper, and I also would like to thank the referee for the detailed corrections.

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